Universität Ulm
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Dr. M. Kunze
M. Gerlach
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Exercises for Applied Analysis

Sheet 13

55. Consider the Hilbert space $L^2(-\pi,\pi)$ with the inner product $(f \mid g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t$ and the orthonormal basis $e_k : t \mapsto e^{ikt}$ for $k \in \mathbb{Z}$.

(a) Compute the Fourier series of the function $f \in L^2(-\pi,\pi)$, $f(t) := |t|$.

(b) Conclude from Parseval's identity, that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$.

56. Consider the Hilbert space $\mathbb{R}^4$ with the cannonical inner product $(x \mid y) = \sum_{k=1}^{4} x_k y_k$ and let $z_1 := (0,1,-1,1)$, $z_2 := (0,1,1,1)$, $z_3 := (2,1,0,2)$, $y := (1,0,0,0)$.

(a) Find an orthonormal basis of $U := \operatorname{span}\{z_1,z_2,z_3\}$.

(b) Determine the distance of y from U, i.e. $\inf\{\|x-y\| : x \in U\}$.

57. On $[0,1]$ consider the functions $f_j(t) = t^j$, $j = 0,1,2$, and let $H := \operatorname{span}\{f_0,f_1,f_2\}$ be the three-dimensional Hilbert space with inner product $(f \mid g) := \int_0^1 f(t)g(t)\,\mathrm{d}t$. The mapping $\varphi : H \to \mathbb{R}$, given by $\varphi(f) := f(1)$, is continuous and linear. Determine an element $h \in H$ such that $\varphi(f) = (f \mid h)$ holds for all $f \in H$.

58. Let $(\Omega,\Sigma,\mathbb{P})$ be a probability space and $(A_n) \subset \Sigma$ be a partition of Ω, i.e. $A_n \cap A_m = \emptyset$ for all $n \neq m$ and $\cup A_n = \Omega$. Let $\mathscr{F} := \sigma(\{A_1,A_2,\dots\})$ denote the generated σ-algebra and $X \in L^2(\Omega,\Sigma,\mathbb{P})$. Determine the conditional expectation of X given $\mathscr{F}$.

Now assume you throw three coins independently and consider the random variables

$$X_j := \begin{cases} 0 & \text{coin } j \text{ shows head} \\ 1 & \text{coin } j \text{ shows tail} \end{cases}$$

for $j = 1,2,3$ and $Y := X_1 + X_2 + X_3$. Determine $\mathbb{E}(Y \mid X_3)$.