55. Consider the Hilbert space $L^2(-\pi, \pi)$ with the inner product $(f \mid g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(t) \, dt$ and the orthonormal basis $e_k : t \mapsto e^{ikt}$ for $k \in \mathbb{Z}$.

(a) Compute the Fourier series of the function $f \in L^2(-\pi, \pi)$, $f(t) := |t|$.

(b) Conclude from Parseval’s identity, that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^4}{96}$.

56. Consider the Hilbert space $\mathbb{R}^4$ with the canonical inner product $(x \mid y) = \sum_{k=1}^{4} x_k y_k$ and let $z_1 := (0, 1, -1, 1)$, $z_2 := (0, 1, 1, 1)$, $z_3 := (2, 1, 0, 2)$, $y := (1, 0, 0, 0)$.

(a) Find an orthonormal basis of $U := \text{span}\{z_1, z_2, z_3\}$.

(b) Determine the distance of $y$ from $U$, i.e. $\inf\{\|x - y\| : x \in U\}$.

57. On $[0, 1]$ consider the functions $f_j(t) = t^j$, $j = 0, 1, 2$, and let $H := \text{span}\{f_0, f_1, f_2\}$ be the three-dimensional Hilbert space with inner product $(f \mid g) := \int_0^1 f(t)g(t) \, dt$. The mapping $\varphi : H \to \mathbb{R}$, given by $\varphi(f) := f(1)$, is continuous and linear. Determine an element $h \in H$ such that $\varphi(f) = (f \mid h)$ holds for all $f \in H$.

58. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $(A_n) \subset \Sigma$ be a partition of $\Omega$, i.e. $A_n \cap A_m = \emptyset$ for all $n \neq m$ and $\cup A_n = \Omega$. Let $\mathcal{F} := \sigma(\{A_1, A_2, \ldots\})$ denote the generated $\sigma$-algebra and $X \in L^2(\Omega, \Sigma, \mathbb{P})$. Determine the conditional expectation of $X$ given $\mathcal{F}$.

Now assume you throw three coins independently and consider the random variables $X_j := \begin{cases} 
0 & \text{coin } j \text{ shows head} \\
1 & \text{coin } j \text{ shows tail} 
\end{cases}$ for $j = 1, 2, 3$ and $Y := X_1 + X_2 + X_3$. Determine $\mathbb{E}(Y \mid X_3)$.

Exercise sheets and further information can be found at:

http://www.uni-ulm.de/mawi/iaa/courses/ws11/applied-analysis.html