Universität Ulm
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Exercises for Applied Analysis

Sheet 2

7. Let (M_j, d_j) be metric spaces for $j \in \{1, \ldots, N\}$. By $M := M_1 \times \cdots \times M_N$ we denote the product space endowed with the product metric (see Proposition 1.1.10). Let $\mathbf{x}_n$, $\mathbf{x} \in M$ for all $n \in \mathbb{N}$, where $\mathbf{x}_n = (x_j^{(n)})$ and $\mathbf{x} = (x_j)$ for $j \in \{1, \ldots, N\}$. Prove that $\mathbf{x}_n \to \mathbf{x}$ in M if and only if $x_j^{(n)} \to x_j$ for all $j \in \{1, \ldots, N\}$ as $n \to \infty$.

Conclude that for every metric space (M', d'), a function $f : M' \to M$ is continuous if and only if every component $f_j : M \to M_j$ is continuous for all $j \in \{1, \ldots, N\}$.

8. Find a sequence of open subsets of $\mathbb{R}$ whose intersection is not open and a sequence of closed subsets of $\mathbb{R}$ whose union is not closed.

9. Consider the set

$$B := \{\mathbf{x} \in \ell^\infty : |x_j| < 1 \text{ for all } j \in \mathbb{N}\}.$$

Decide whether B is open in ℓ^∞ if ℓ^∞ is endowed with

(a) the metric d_0.

(b) the metric induced by the norm $\|\cdot\|_\infty$.

10. Consider the set

$$A := (\{(x, y) \in \mathbb{R}^2 : y < 0\} \backslash \{(1/n, -1) : n \in \mathbb{N}\}) \cup \{(1/n, 1) : n \in \mathbb{N}\}.$$

and determine the sets A°, $\bar{A}$ and ∂A as well as the set of all accumulation points of A.

11. We consider the metric spaces $(\mathbb{R}, d)$ and $(\mathbb{R}, d')$, where d denotes the discrete and d' the euclidean metric.

(a) Describe the convergent sequences of $(\mathbb{R}, d)$.

(b) Determine all continuous functions $f : (\mathbb{R}, d) \to (\mathbb{R}, d')$.

12. Let E be a vector space over $\mathbb{K}$ and $\|\cdot\|_1$, $\|\cdot\|_2$ be norms on E. Assume that a sequence in E converges to 0 with respect to $\|\cdot\|_1$ if and only if it converges to 0 with respect to $\|\cdot\|_2$. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

13. Let (M, d) and (M', d') be two metric spaces such that (M, d) is separable. Show that (M', d') is also separable if there exists a surjective continuous mapping $f : M \to M'$.