Exercises for Applied Analysis

Sheet 4

18. Let (M,d) be a metric space. Prove that the following assertions are equivalent.

(i) (M,d) is compact.

(ii) For every family $(A_j)_{j\in J}$ of closed subsets of M (for some index set J) satisfying $\cap_{j\in J} A_j = \emptyset$ it follows that $\cap_{j\in F} A_j = \emptyset$ for some finite set $F \subset J$.

19. Let (M_1, d_1) and (M_2, d_2) be metric spaces and $f : M_1 \to M_2$ a continuous function. Prove that $f(K)$ is compact for all compact sets $K \subset M_1$.

Conversely, let $g : M_1 \to M_2$ be a function such that $g(K)$ is compact for all compact sets $K \subset M_1$. Does this imply that g is continuous?

20. Let $1 \le p < \infty$ and $\mathbf{x} = (x_n) \in \ell^p$ such that $x_n \ge 0$ for all $n \in \mathbb{N}$. We denote by

$$[0,\mathbf{x}] := \{\mathbf{y} = (y_n) \in \ell^p : 0 \le y_n \le x_n \ \forall n \in \mathbb{N}\},$$

a so-called *order intervall*. Prove that $[0,\mathbf{x}]$ is compact in ℓ^p.

21. Give one example each for the following situations.

(a) An unbounded and continuous function $f : U \to \mathbb{R}$ on a bounded interval $U \subset \mathbb{R}$.

(b) A bounded function $f : M \to \mathbb{R}$ on a compact metric space (M,d) which does not have a maximum.

(c) A bounded but not totally bounded subset of a normed space.

(d) An open cover of $\{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$, endowed with the euclidean metric, which does not have a finite subcover.

22. Decide which of the following sets $\mathcal{A} \subset C(M)$ are unital algebras, which separate the points in M and which are closed under conjugation.

(a) We endow $M := \{z \in \mathbb{C} : |z| = 1\}$ with the euclidean metric d. Let $\mathcal{A}$ be the set of all complex polynomials on M, i.e., $\mathcal{A} := \{z \mapsto \sum_{k=0}^n a_k z^k : n \in \mathbb{N},\ a_k \in \mathbb{C}\}$.

(b) We endow $M := \{z \in \mathbb{C} : |z| = 1\}$ with the euclidean metric d. Let $\mathcal{A}$ be the set of all complex polynomials on M in the variables z and $\frac{1}{z}$, i.e., $\mathcal{A} := \{z \mapsto \sum_{k=-n}^n a_k z^k : n \in \mathbb{N},\ a_k \in \mathbb{C}\}$.

(c) Consider $M := [0,1] \times [0,1]$ with the euclidean metric d. Let $\mathcal{A}$ be the set of all polynomials on M, i.e., $\mathcal{A} := \{(x,y) \mapsto \sum_{k,l=0}^n a_{k,l} x^k y^l : n \in \mathbb{N},\ a_{k,l} \in \mathbb{R}\}$.

(d) Consider $M := [0,1]$ with the euclidean metric d. Let $\mathcal{A}$ be the set of all polynomials on M with even exponents only, i.e., $\mathcal{A} := \{x \mapsto \sum_{k=0}^n a_k x^{2k} : n \in \mathbb{N},\ a_k \in \mathbb{R}\}$.

(e) Consider $M := [0,1]$ with the euclidean metric d. Let $\mathcal{A}$ be the set of all polynomials on M with odd exponents only, i.e., $\mathcal{A} := \{x \mapsto \sum_{k=0}^n a_k x^{2k+1} : n \in \mathbb{N},\ a_k \in \mathbb{R}\}$.