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Exercises for Applied Analysis

Sheet 5

23. Endow $M := [1, \infty)$ with the euclidean metric and consider the mapping $\varphi : M \to M$ given by $\varphi(x) := x + \frac{1}{x}$. Show that

$$|\varphi(x) - \varphi(y)| < |x - y|$$

for all $x, y \in M$ and that φ does not have a fixed point.

Why does this not conflict with Banach's fixed point theorem?

24. Let $T > 0$ and consider the ordinary differential equation

$$\text{(P)} \quad \begin{cases} u'(t) = 2tu(t) & t \in [0, T] \\ u(0) = 1 \end{cases}$$

(see Definition 1.8.7).

(a) Prove that there exists a unique solution $u^* : [0, T] \to \mathbb{R}$ of (P).

(b) Let $\varphi : C([0, T]) \to C([0, T])$ be corresponding to (P) as in Lemma 1.8.8. and let $v : C([0, T]) \to C([0, T])$ be given by $v(t) := 1$ for all $t \in [0, T]$. Compute $\varphi(v)$, $\varphi^2(v)$ and $\varphi^3(v)$.

(c) Guess the solution of (P).

25. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{A} = \{\{1, 2\}, \{2, 3, 4\}\}$. Determine $\sigma(\mathcal{A})$.

26. Let $\Omega = \mathbb{N}$ and denote by $2\mathbb{N} = \{2k : k \in \mathbb{N}\}$ the set of all even numbers. Determine $\sigma(\mathcal{P}(2\mathbb{N}))$.

27. Consider the metric space $M = \mathbb{R}$ endowed with the euclidean metric d. Prove that $\mathscr{B}(M, d) = \sigma(\mathcal{A})$ where

(a) $\mathcal{A} = \{(a, b) : a, b \in \mathbb{R},\ a < b\}$.

(b) $\mathcal{A} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.

(c) $\mathcal{A} = \{(a, \infty) : a \in \mathbb{R}\}$.

28. Let (M, d) be a separable metric space. Prove that $\mathscr{B}(M, d) = \sigma(\mathcal{A})$ where

$$\mathcal{A} = \{\bar{B}(x, r) : x \in M, r > 0\}.$$