29. Let $(\Omega_j, \Sigma_j)$ be measurable spaces for $j = 1, 2, 3$. Let $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_2 \rightarrow \Omega_3$ be measurable functions. Prove that $g \circ f : \Omega_1 \rightarrow \Omega_3$ is measurable.

30. Give one example each for the following situations.

(a) A measure space $(\Omega, \Sigma, \mu)$ and a sequence $(A_n) \subset \Sigma$ such that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ but $\mu(A_n) \not\rightarrow \mu(\bigcap_{k \in \mathbb{N}} A_k)$ as $n \to \infty$.

(b) A measure space $(\Omega, \Sigma, \mu)$ with a generator $A \subset \Sigma$ such that $\mu(A) = 0$ for all $A \in A$ but $\mu(\Omega) \not= 0$.

31. Let $\Omega := \{0, 1\} \times \{0, 1\}$ and let $f_1, f_2 : \Omega \rightarrow \mathbb{R}$ be given by

$$f_1(x, y) := x + y \quad \text{and} \quad f_2(x, y) := y.$$ 

Throughout, $\mathbb{R}$ is endowed with its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$.

(a) Determine $\Sigma_1 := \sigma(\{f_1\})$, $\Sigma_2 := \sigma(\{f_1, f_2\})$ and $\Sigma_3 := \mathcal{P}(\{0, 1\}) \otimes \{\emptyset, \{0, 1\}\}$.

(b) Decide whether the function $g : \Omega \rightarrow \mathbb{R}$, given by $g(x, y) := x$, is $\Sigma_j/\mathcal{B}(\mathbb{R})$-measurable for $j = 1, 2, 3$.

32. Let $\Omega = C([0, 1])$, the continuous function from $[0, 1]$ to $\mathbb{R}$, and consider the mappings $\pi_t : \Omega \rightarrow \mathbb{R}$, given by $\pi_t(f) := f(t)$. Throughout, $\mathbb{R}$ is endowed with its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$. We define $\mathcal{F}_t := \sigma(\{\pi_s : 0 \leq s \leq t\})$ and $\mathcal{F}_{t+} := \cap_{s \geq t} \mathcal{F}_s$ for all $t \geq 0$.

Now fix some $0 < t < 1$ and let $A := \{f \in \Omega : f$ has a local maximum point at $t\}$.

(a) Prove that

$$A = \bigcup_{m=1}^{\infty} \bigcap_{r \in \mathbb{Q} \cap (-1/m, 1/m)} \{f \in \Omega : f(t) \geq f(r)\}$$

for all $n \in \mathbb{N}$ such that $t - \frac{1}{n} > 0$ and $t + \frac{1}{n} < 1$.

(b) Conclude from the previous part that $A \in \mathcal{F}_{t+}$.

**Hint:** You may use without proving it that $\pi_r - \pi_t$ is $\mathcal{F}_{t+1/n}/\mathcal{B}(\mathbb{R})$-measurable for all $r \leq t + \frac{1}{n}$.

(c) Let $B \in \mathcal{F}_t$ and $f \in B$. Show that every function $g \in \Omega$ that satisfies $g(s) = f(s)$ for all $s \leq t$ belongs to $B$, too.

**Hint:** Use the principle of good sets.

(d) Conclude from the previous part that $A \not\in \mathcal{F}_t$.