UNIVERSITÄT ULM
Discussion of solutions:
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Exercises for Applied Analysis

Sheet 6

29. Let (Ω_j, Σ_j) be measurable spaces for $j = 1, 2, 3$. Let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ be measurable functions. Prove that $g \circ f : \Omega_1 \to \Omega_3$ is measurable.

30. Give one example each for the following situations.

(a) A measure space (Ω, Σ, μ) and a sequence $(A_n) \subset \Sigma$ such that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ but

$$\mu(A_n) \not\to \mu\Big(\bigcap_{k\in\mathbb{N}} A_k\Big) \quad \text{as} \quad n \to \infty.$$

(b) A measure space (Ω, Σ, μ) with a generator $\mathcal{A} \subset \Sigma$ such that $\mu(A) = 0$ for all $A \in \mathcal{A}$ but $\mu(\Omega) \neq 0$.

31. Let $\Omega := \{0, 1\} \times \{0, 1\}$ and let $f_1, f_2 : \Omega \to \mathbb{R}$ be given by

$$f_1(x, y) := x + y \qquad \text{and} \qquad f_2(x, y) := y.$$

Throughout, $\mathbb{R}$ is endowed with its Borel σ-algebra $\mathscr{B}(\mathbb{R})$.

(a) Determine $\Sigma_1 := \sigma(\{f_1\})$, $\Sigma_2 := \sigma(\{f_1, f_2\})$ and $\Sigma_3 := \mathscr{P}(\{0, 1\}) \otimes \{\emptyset, \{0, 1\}\}$.

(b) Decide whether the function $g : \Omega \to \mathbb{R}$, given by $g(x, y) := x$, is $\Sigma_j/\mathscr{B}(\mathbb{R})$-measurable for $j = 1, 2, 3$.

32. Let $\Omega = C([0, 1])$, the continuous function from $[0, 1]$ to $\mathbb{R}$, and consider the mappings $\pi_t : \Omega \to \mathbb{R}$, given by $\pi_t(f) := f(t)$. Throughout, $\mathbb{R}$ is endowed with its Borel σ-algebra $\mathscr{B}(\mathbb{R})$. We define $\mathscr{F}_t := \sigma(\{\pi_s : 0 \le s \le t\})$ and $\mathscr{F}_{t+} := \cap_{s>t}\mathscr{F}_s$ for all $t \ge 0$.

Now fix some $0 < t < 1$ and let $A := \{f \in \Omega : f \text{ has a local maximum point at } t\}$.

(a) Prove that

$$A = \bigcup_{m=n}^{\infty} \bigcap_{\substack{r\in\mathbb{Q} \\ |t-r|<1/m}} \{f \in \Omega : f(t) \ge f(r)\}$$

for all $n \in \mathbb{N}$ such that $t - \frac{1}{n} > 0$ and $t + \frac{1}{n} < 1$.

(b) Conclude from the previous part that $A \in \mathscr{F}_{t+}$.

Hint: You may use without proving it that $\pi_r - \pi_t$ is $\mathscr{F}_{t+1/n}/\mathscr{B}(\mathbb{R})$-measurable for all $r \le t + \frac{1}{n}$.

(c) Let $B \in \mathscr{F}_t$ and $f \in B$. Show that every function $g \in \Omega$ that satisfies $g(s) = f(s)$ for all $s \le t$ belongs to B, too.

Hint: Use the principle of good sets.

(d) Conclude from the previous part that $A \notin \mathscr{F}_t$.