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Exercises for Applied Analysis

Sheet 7

33. Let μ be a probability measure on $\mathscr{B}(\mathbb{R})$. The cumulative distribution function $F_\mu : \mathbb{R} \to \mathbb{R}$ is defined by

$$F_\mu(t) := \mu((-\infty, t]) \quad (t \in \mathbb{R}).$$

(a) Show that F_μ has the following properties.

(i) F_μ is monotonically increasing.

(ii) F_μ is right continuous, i.e. $\lim_{n\to\infty} F_\mu(t_n) = F_\mu(t)$ whenever $\lim_{n\to\infty} t_n = t$ and $t_n \geq t$ for all $n \in \mathbb{N}$.

(iii) $\lim_{t\to-\infty} F_\mu(t) = 0$.

(iv) $\lim_{t\to\infty} F_\mu(t) = 1$.

(v) F_μ is continuous if and only if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$.

(b) Let ν be another probability measure on $\mathcal{B}(\mathbb{R})$. Prove that $\mu = \nu$ if and only if $F_\mu = F_\nu$.

(c) Determine the measure μ if the cumulative distribution function is given as

(i) $F_\mu(t) = \mathbb{1}_{[x_0,\infty)}(t)$.

(ii) $F_\mu(t) = t \cdot \mathbb{1}_{[0,1]}(t) + \mathbb{1}_{(1,\infty)}(t)$.

34. Let λ be the Lebesgue measure on $\mathscr{B}(\mathbb{R})$. Prove that $\lambda(A) = \lambda(-A)$ for all $A \in \mathscr{B}(\mathbb{R})$ where $-A := \{-x : x \in A\}$.

35. Let Ω be a set. For a sequence $(A_n) \subset \mathscr{P}(\Omega)$ of subsets of Ω we define

$$\limsup(A_n) := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

and

$$\liminf(A_n) := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

A sequence $(A_n) \subset \mathscr{P}(\Omega)$ is said to be *convergent to some* $A \subset \Omega$ if

$$A = \liminf(A_n) = \limsup(A_n).$$

(a) Let (Ω, Σ, μ) be a finite measure space. Show that $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ for every sequence $(A_n) \subset \Sigma$ that converges to $A \in \Sigma$.

(b) For $n \in \mathbb{N}$ let $A_n := [-1/n, 1/n] \times [-n, n] \in \mathscr{B}(\mathbb{R}^2)$. Prove that (A_n) converges to $\{0\} \times \mathbb{R}$, compute $\lambda_2(\{0\} \times \mathbb{R})$ and conclude that (a) does not hold in infinite measure spaces.

(c) Let (Ω, Σ, μ) be a measure space and $(A_n) \subset \Sigma$ such that $\sum_{n=1}^\infty \mu(A_n) < \infty$. Show that

$$\mu(\{x \in \Omega : x \in A_n \text{ for intinitely many } n \in \mathbb{N}\}) = 0.$$