Universität Ulm
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Exercises for Applied Analysis

Sheet 8

36. Consider the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \zeta)$ where ζ is the counting measure on $\mathscr{P}(\mathbb{N})$. Prove that a function $f : \mathbb{N} \to \mathbb{R}$ is integrable if and only if the sequence $(f(k))$ belongs to ℓ^1. Moreover, show that in this case

$$\int_{\mathbb{N}} f \, \mathrm{d}\zeta = \sum_{k=1}^{\infty} f(k).$$

37. Let (Ω, Σ, μ) be a measure space and $h : \Omega \to [0, \infty)$ be a measurable function. We define the mapping $h\mu : \Sigma \to [0, \infty]$ by

$$(h\mu)(A) := \int_{\Omega} \mathbb{1}_A h \, \mathrm{d}\mu$$

(a) Show that $h\mu$ is a measure on Σ.

(b) For a measurable function $f : \Omega \to \mathbb{R}$ prove that $f \in \mathscr{L}^1(\Omega, \Sigma, h\mu)$ if and only if $fh \in \mathscr{L}^1(\Omega, \Sigma, \mu)$. Moreover, show that in this case

$$\int_{\Omega} f \, \mathrm{d}(h\mu) = \int_{\Omega} fh \, \mathrm{d}\mu.$$

38. Determine in which of the following situations the limit

$$\lim_{n\to\infty} \int_{\Omega} f_n \, \mathrm{d}\mu$$

exists. If so, compute its value.

(a) Let $\Omega := [0, 2\pi]$, $\mu := \lambda$ and $f_n(x) := \sin(x)^n$.

(b) Let $\Omega := [0, 2\pi]$, $\mu := \delta_{3\pi/2}$ and $f_n(x) := \sin(x)^n$.

(c) Let $\Omega := \mathbb{R}$, $f_n := \sum_{k=1}^{n} k \mathbb{1}_{[k-1,k)}$ and μ be the unique Borel measure whose c.d.f. is given by

$$F_\mu(t) = \begin{cases} 0 & t \le 0 \\ 1 - \exp(-t) & t > 0 \end{cases}.$$

(d) Let $\Omega := [0, 2]$, $\mu := \lambda$ and $f_n(x) := x^n$.

(e) Let $\Omega := (0, \infty)$, $f_n(x) := (1 - x^{-n})\mathbb{1}_{[1,n]}(x)$ and μ be the push-forward of λ under $\Phi : (0, \infty) \to (0, \infty)$, $\Phi(x) := \frac{1}{x}$.

(f) Let $\Omega := \mathbb{N}$, $f_n(k) := (\frac{k}{n} - 2)^k \mathbb{1}_{\{1,\dots,n\}}(k)$ and $\mu := h\zeta$ where ζ denotes the counting measure and $h : \mathbb{N} \to \mathbb{N}$ ist given by $h(k) := \frac{1}{4^k}$.

In (a) – (e), Ω is endowed with its Borel-σ-algebra. In (f) we endow Ω with $\mathscr{P}(\mathbb{N})$.