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Exercises for Applied Analysis

Sheet 9

39. Let $p \in [1, \infty]$. Determine for which $\alpha > 0$ the function

(a) $f : [1, \infty) \to \mathbb{R}$, $f(t) := t^{-\alpha}$, belongs to $\mathscr{L}^p([1, \infty), \mathscr{B}, \lambda)$.

(b) $f : (0, 1] \to \mathbb{R}$, $f(t) := t^{-\alpha}$, belongs to $\mathscr{L}^p((0, 1]), \mathscr{B}, \lambda)$.

40. Consider the measure space $([0, 1], \mathscr{B}, \lambda)$

(a) Let $f : [0, 1] \to \mathbb{R}$ be measurable. Show that $[f]$ is uncountable.

(b) Let $f, g : [0, 1] \to \mathbb{R}$ be continuous. Prove that $f(x) = g(x)$ for all $x \in [0, 1]$ whenever $f(x) = g(x)$ for almost every $x \in [0, 1]$. This shows f is the only continuous function in $[f]$.

Hint: Use that the set $\{f = g\}$ is closed.

41. We denote by

$$L^p_+ := \{f \in L^p(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda) : f(x) \geq 0 \text{ almost everywhere } \}$$

the *positive cone* of $L^p(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

(a) Use Lemma 2.7.8 to show that L^p_+ is closed.

(b) Show that the interior of L^p_+ is empty.

42. Give one example each for the following situations.

(a) A function $f \in L^1(\mathbb{R}, \mathscr{B}, \lambda)$ that is not essentially bounded near infinity. Here, we call a function $f : \mathbb{R} \to \mathbb{R}$ *essentially bounded near infinity*, if there exists $C > 0$ and $n_0 \in \mathbb{N}$ such that $|f(t)| \leq C$ almost everywhere whenever $|t| \geq n_0$.

(b) A sequence of functions $f_n \in L^2((0, 1), \mathscr{B}, \lambda)$ such that $f_n \to 0$ in $L^p(0, 1)$ for all $1 \leq p < 2$ but $f_n \not\to 0$ in $L^2(0, 1)$.

(c) A measure μ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that for every measurable function $f : \mathbb{R} \to \mathbb{R}$ there exists $c \in \mathbb{R}$ such that $[f] = [c\mathbb{1}]$, i.e. that $L^p(\mathbb{R}, \mathscr{B}, \mu)$ is one-dimensional for every $1 \leq p \leq \infty$.

(d) A measure space (Ω, Σ, μ) such that $L^p(\Omega, \Sigma, \mu) \subset L^q(\Omega, \Sigma, \mu)$ for $1 \leq p < q \leq \infty$.

Merry Christmas and a Happy New Year!