Applied Analysis

Testexam

Name: ____________________________

	Part I	Part II	Part III				Σ
Maximal points	40	5+5+5+5	10	10	10	10	100
Points reached							

Grade: ___________

- Time limit: 150 min. (2h 30min)
- Allowed aids for the exam: 1 sheet of paper (DIN A4) upon which you may write (by hand!) whatever you find useful. Noting else. (In particular: *no calculator*).
- If you want your exam graded, hand it in during the lecture on Tuesday, Dec 20th. Note that the outcome of this testexam has no influence whatsoever on your final grade.

Part I: Multiple choice (40 Points)

In the following multiple choice questions, there is a total of 20 correct answers. However, for each individual question, there may be more than one correct answer or no correct answer at all.

Cross the boxes that correspond to correct answers. For each correctly placed cross, you get 2 points. For an incorrectly placed cross, you get 0 points. At most 20 crosses are graded. If you place more than 20 crosses, incorrectly placed crosses are taken into account first.

(1) Let (M, d) be a metric space, $A \subset M$ be such that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ which converges to a point $x_0 \in M \setminus A$. Then:

- ☐ A is dense in M.
- ☐ $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
- ☐ $x_0 \in \overline{A}$.
- ☐ (M, d) is not complete.

(2) Consider the set $A := \{(x, y) \in \mathbb{R}^2 : 0 < x, y < 1\} \subset \mathbb{R}^2$. Then

- ☐ A is an open ball in $(\mathbb{R}^2, \|\cdot\|_\infty)$.
- ☐ A is open in $(\mathbb{R}^2, \|\cdot\|_2)$.
- ☐ In $(\mathbb{R}^2, \|\cdot\|_1)$, the set A contains all its accumulation points.
- ☐ In $(\mathbb{R}^2, \|\cdot\|_6)$, every point in A is an accumulation point of A.

(3) Let (M_j, d_j) be metric spaces for $j = 1, 2$ and $f : M_1 \to M_2$ be continuous. Then:

- ☐ $f(U)$ is open in M_2, whenever U is open in M_1.
- ☐ $f^{-1}(K)$ is compact in M_1 for all compact $K \subset M_2$.
- ☐ $f(K)$ is compact in M_2 for all compact $K \subset M_1$.
- ☐ If (M_1, d_1) is complete, then so is (M_2, d_2).

(4) In a compact metric space

- ☐ every sequence is a Cauchy sequence.
- ☐ every Cauchy sequence converges.
- ☐ every sequence has a subsequence which is a Cauchy sequence.
- ☐ every sequence which has a convergent subsequence is a Cauchy sequence.

(5) Let (M, d) be a metric space, $A \subset M$ be closed and $B \subset M$ be open. Then

- ☐ $A \neq B$.
- ☐ $A \cap B$ is neither open nor closed.
- ☐ $A \cup B$ is neither open nor closed.
- ☐ $B \subset A$.

(6) Consider the normed space $(C([0, 1]), \|\cdot\|_\infty)$.

- ☐ $(C([0, 1]), \|\cdot\|_\infty)$ is separable.
- ☐ Every map from $(C([0, 1]), \|\cdot\|_\infty)$ into itself has a fixed point.
- ☐ Every closed and bounded subset of $(C([0, 1]), \|\cdot\|_\infty)$ is compact.
- ☐ $(C([0, 1]), \|\cdot\|_\infty)$ is complete.

(7) Let (Ω, Σ) be a measurable space, I be a set and $A_i \in \Sigma$ for all $i \in I$. Then $\bigcup_{i \in I} A_i \in \Sigma$ whenever

- ☐ $I = [0,1]$.
- ☐ $I \in \mathscr{B}(\mathbb{R})$.
- ☐ I is finite.
- ☐ $I = \mathbb{Q}$.

(8) Consider the measure space

$$(\Omega, \Sigma, \mu) = (\{1,2,3,4\}, \sigma(\{\{1,2\},\{2,3,4\}\}), \delta_2 + 2\delta_3).$$

Then:

- ☐ $\{1\} \in \Sigma$.
- ☐ $\{4\} \in \Sigma$.
- ☐ $\{4\}$ is a null set.
- ☐ $\int_\Omega 3\mathbb{1}_{\{1,2\}} - \mathbb{1}_{\{2,3,4\}} \, d\mu = 0$.

(9) The Lebesgue measure of the set $\{\pi\}$ is

- ☐ 0.
- ☐ 1.
- ☐ π.
- ☐ not defined, since $\{\pi\} \notin \mathscr{B}(\mathbb{R})$.

(10) Let f, g, h be functions from a set Ω to $\mathbb{R}$, which is endowed with its Borel σ-algebra. Let $\Sigma = \sigma(\{f+g, g+h\})$. Which of the following functions are $\Sigma/\mathscr{B}(\mathbb{R})$-measurable?

- ☐ f.
- ☐ g.
- ☐ $f - h$.
- ☐ $\varphi : \Omega \to \mathbb{R}$, defined by $\varphi(x) = \cos(f(x) + g(x)) + 5$.

(11) Let μ be the probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with cumulative distribution function

$$F_\mu(t) = \frac{1}{2}\mathbb{1}_{[1,3)} + \mathbb{1}_{[3,\infty)}.$$

Then

- ☐ $\mu(\mathbb{Q}) = 0$.
- ☐ $\mu((1,3)) = \frac{1}{2}$.
- ☐ $\mu((-\sqrt{2}, 4)) = 1$.
- ☐ $\{-5\} \cup [18, 22)$ is a null set.

(12) Let (Ω, Σ, μ) be a measure space and f_n be a sequence of positive, measurable functions such that $f_n \downarrow 0$ but $\int_\Omega f_n \, d\mu$ does not converge to 0. Then

- ☐ μ is not σ-finite.
- ☐ none of the functions f_n is integrable.
- ☐ μ is not finite.
- ☐ $\lim_{n\to\infty} f_n$ is not integrable.

Part II: Examples (20 points)

In this part, there are 8 situations described. These situations are grouped into two sets ("Set A" and "Set B"). From each set, pick exactly two situations. Then give a specific example for each situation you have picked. Indicate which situation (e.g. (A2), (B4), ...) your example refers to. Always explain your example (e.g. in (A1), explain why the vector space you name is not complete, in (A2), explain why the set you name is an algebra and why it separates points, etc.)
Each of your example is worth up to 5 points; you may recieve partial credit.

Set A

(A1) A normed vector space which is not complete.
(A2) An algebra of real-valued, continuous functions on $[0,1]$, different from all of $C([0,1])$, which separates points.
(A3) A subset A of a metric space, such that all of the sets $A, \overline{A}, A^\circ, \overline{A}^\circ, \overline{A^\circ}$ are different.
(A4) A function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous and an open subset U of $\mathbb{R}$ such that $f^{-1}(U)$ is not open.

Set B

(B1) A measurable space (Ω, Σ) and two functions $f, g : \Omega \to \mathbb{R}$ which are not measurable such that $f + g$ is measurable.
(B2) A measure space (Ω, Σ, μ) which is not σ-finite.
(B3) A measure space (Ω, Σ, μ) and a sequence of integrable functions $f_n : \Omega \to \mathbb{R}$, converging to an integrable function $f : \Omega \to \mathbb{R}$ such that $\int_\Omega f_n \, d\mu$ does not converge to $\int_\Omega f \, d\mu$.
(B4) Two measures μ, ν on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and a measurable function $f : \mathbb{R} \to [0, \infty)$ such that $f \in \mathscr{L}^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu)$ and $f \notin \mathscr{L}^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \nu)$.

Write your examples below this line (you may also use the back of this page).

Part III: Problems (40 points)

In what follows, you are given 4 problems. Solve each of them, showing all your work. Each problem is worth 10 points. You may recieve partial credit, however, this part of the exam is focused on your arguing, so make sure to explain your reasoning.

(Problem 1)

Let $\varphi : C([0,1]) \to C([0,1])$ be defined by

$$[\varphi(f)](t) := 1 + \frac{1}{2}\sin(f(1)) - \int_0^t \frac{f(s/2)+1}{4}\, ds\,.$$

Prove that φ has a unique fixed point.

Write your answer below this line (you may also use the back of this page).

(Problem 2)

In the metric space $(\ell^\infty, \|\cdot\|_\infty)$, consider the set

$$A := \{\mathbf{x} \in \ell^\infty \,:\, 0 < x_k < 1 \text{ for all } k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} x_k < 1\}\,.$$

Prove that $A^\circ = \emptyset$.

Write your answer below this line (you may also use the back of this page).

(Problem 3)

On $\mathbb{R}$, let $\Sigma := \sigma(\{(-b,-a) \cup (a,b) \,:\, 0 < a < b\})$. Show that $\mathbb{Z} \in \Sigma$, where $\mathbb{Z}$ is the set of integers $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

Write your answer below this line (you may also use the back of this page).

(Problem 4)

On $([0,2], \mathscr{B}([0,2]), \lambda)$, let $f_n : [0,2] \to \mathbb{R}$ be given by

$$f_n(x) = \frac{x^n}{1+x^n}\,.$$

Decide, whether the sequence $\int_{[0,2]} f_n \, d\lambda$ converges. If so, determine its limit.

Write your answer below this line (you may also use the back of this page).

If for some of the problems of parts II or III you need additional space, complete your answers below or on the following pages. Indicate clearly which problem you are referring to.