Positive Semigroups and Perron Frobenius Theory

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Theorem

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(i) $r(T) \in \sigma(T)$.

(ii) $|\lambda| < r(T)$ *for all other eigenvalues* $\lambda \in \sigma(T)$.

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Example

- Let T be the transition matrix of a Markov chain on a finite state space.
- If all transition probabilities $t_{kl} > 0$, then the Theorem can be applied to analyse the spectrum of T.

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$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{i.e.} \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}.$$

Then $r(T) = 1$, $\sigma(T) = \{1, e^{\frac{1}{3}2\pi i}, e^{\frac{2}{3}2\pi i}\}$.

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However:

Theorem

Let $T \in \mathbb{R}^{d\times d}$ with non-negative entries $t_{kl} \geq 0$. Then:

(i) $r(T) \in \sigma(T)$.

(ii) *Whenever $r(T)e^{i\varphi} \in \sigma(T)$, then $r(T)e^{in\varphi} \in \sigma(T)$ for all $n \in \mathbb{Z}$.*

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Definition

A (bounded) linear operator $T : \mathbb{X} \to \mathbb{X}$ is called **positive** if it maps positive elements to positive elements.

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Let $T : \mathbb{X} \to \mathbb{X}$ be a positive linear operator. Then

(i) $r(T) \in \sigma(T)$.

Furthermore, suppose that $||T^n|| \leq M\, r(T)^n$ for an $M \geq 0$ and all $n \in \mathbb{N}_0$. Then we also have:

(ii) *Whenever $r(T)e^{i\varphi} \in \sigma(T)$, then $r(T)e^{in\varphi} \in \sigma(T)$ for all $n \in \mathbb{Z}$.*

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Remark

The powers $\{T^n, n \in \mathbb{N}_0\}$ *of an operator* T *form a* **discrete operator semigroup**, *i.e.* $T^{n_1+n_2} = T^{n_1} T^{n_2}$.

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This continuous analogue of the notion is:

Definition

A C_0-**semigroup** $(T(t))_{t\geq 0}$ on $\mathbb{X}$ is a strongly continuous mapping $T : [0, \infty) \to \mathcal{L}(\mathbb{X})$ such that

$$T(0) = \mathrm{Id}_{\mathbb{X}}, \quad T(t_1 + t_2) = T(t_1)T(t_2).$$

A C_0-semigroup is called positive, if each operator $T(t)$ is positive.

Remark

The set of all $x \in \mathbb{X}$ such that the limit $Ax := \lim_{t\to 0} \frac{T(t)x - T(0)x}{t}$ exists, is a dense linear subspace of $\mathbb{X}$.
The operator A defined on this space is called the **infinitesimal generator** *of the C_0-semigroup.*

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Note: Semigroups are a generalization of the matrix exponential function $T(t) = e^{tA}$.

Theorem

Let $(T(t))_{t\geq 0}$ *be positive* C_0*-semigroup on* $\mathbb{X}$*.*

(i) *We have* $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \in \sigma(A)$*.*

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(i) *We have $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \in \sigma(A)$.*

Furthermore, suppose that $||T(t)|| \leq M\, e^{s(A)\,t}$ for a constant $M \geq 0$ and all $t \in [0, \infty)$. Then we also have:

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Example

Let $\mathbb{X} = L^2(\mathbb{R})$ and let $(T(t))_{t\geq 0}$ be the left shift semigroup, i.e. $T(t)f(x) = f(x+t)$. One can prove that

- A is defined on the Sobolev space $H^1(\mathbb{R})$ and $Af = f'$ for all $f \in H^1(\mathbb{R})$.
- $s(A) = 0$ and $\sigma(A) = i\mathbb{R}$.

Eventually positive operators and semigroups:

- An operator $T \in \mathcal{L}(\mathbb{X})$ is **eventually positive**, if T^n is positive for all $n \geq n_0$.
- A C_0-semigroup $(T(t))_{t \geq 0}$ is **eventually positive**, if $T(t)$ is positive for all $t \geq t_0$.

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Perron Frobenius Theory for eventually positive operators and semigroups:

- Results for the finite dimensional case were recently obtained, e.g. in [1] and [2].

[1] D. Noutsos, *On Perron-Frobenius property of matrices having some negative entries*, Linear Algebra Appl., 412 (2005), 132–153.

[2] D. Noutsos and M. Tsatsomeros, *Reachability and holdability of nonnegative states*, SIAM Journal on Matrix Analysis and Applications, 30 (2008), 700–712.

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- Currently: Investigation of the infinite dimensional case.

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