Eventual Positivity of Operator Semigroups

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based on joint work with W. Arendt, D. Daners and J. Kennedy

Assumptions throughout the talk:

(i) Let E be a complex Banach lattice, e.g. $E = C(K)$ for a compact space K, or $E = L^p(\Omega, \Sigma, \mu)$.

(ii) Let $(e^{tA})_{t \geq 0}$ be a C_0-semigroup on E.

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Definition

The semigroup $(e^{tA})_{t \geq 0}$ is called...

(i) ...*positive*, if $e^{tA}x \geq 0$ for all $x \geq 0$ and for all $t \geq 0$.

(ii) ...*uniformly eventually positive* if there is a $t_0 \in [0, \infty)$ such that $e^{tA}x \geq 0$ for all $x \geq 0$ and for all $t \geq t_0$.

(iii) ...*individually eventually positive* if for each $x \geq 0$ there is a $t_0 \in [0, \infty)$ such that $e^{tA}x \geq 0$ whenever $t \geq t_0$.

Example

Let $E = \mathbb{C}^3$ and let $\mathcal{B} = (u_1, u_2, u_3)$ be the orthonormal basis given by

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

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Let the representation matrix of e^{tA} with respect to the basis $\mathcal{B}$ be given by

$$\exp(t \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} \cos t & -e^{-t} \sin t \\ 0 & e^{-t} \sin t & e^{-t} \cos t \end{pmatrix}.$$

Then $(e^{tA})_{t \geq 0}$ is individually eventually positive.

Remark

Let $E = \mathbb{C}^n$ *and let* $(e^{tA})_{t \geq 0}$ *be individually eventually positive. For large* t*, we have* $e^{tA}e_1 \geq 0$*, ...,* $e^{tA}e_n \geq 0$*.*
Thus, e^{tA} *is uniformly eventually positive.*

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Let $E = C([-1,1])$ and $F := \{f \in E : \int f \, d\lambda = 0\}$. Then $E = \langle \mathbb{1} \rangle \oplus F$.

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Let $E = C([-1,1])$ and $F := \{f \in E : \int f \, d\lambda = 0\}$. Then $E = \langle \mathbb{1} \rangle \oplus F$.
Let R be the reflection operator on F, i.e.

$$Rf(\omega) = f(-\omega) \quad \text{for all } f \in E \text{ and for all } \omega \in [-1,1].$$

Then $\sigma(R) = \{-1, 1\}$.

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$$Rf(\omega) = f(-\omega) \quad \text{for all } f \in E \text{ and for all } \omega \in [-1,1].$$

Then $\sigma(R) = \{-1, 1\}$. The operator

$$A = 0_{\langle \mathbb{1} \rangle} \oplus (R - 2\,\mathrm{id}_F)$$

generates an individually eventually positive semigroup on E.

The following theorem is well-known for positive semigroups.

Theorem

Let $(e^{tA})_{t\geq 0}$ be individually eventually positive with growth bound ω and spectral bound $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$.

(i) *We always have $s(A) \in \sigma(A)$.*

(ii) *If $E = C(K)$ or $E = L^1(\Omega, \Sigma, \mu)$ or E is a Hilbert space, then $s(A) = \omega$.*

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Question

For positive semigroups, (ii) is also true on $E = L^p(\Omega, \Sigma, \mu)$ and on $E = C_0(L)$ for a locally compact space L.
Does this remain true for (individually or uniformly) eventually positive semigroups?

Let $E = C(K)$.

(i) We write $f > 0$ if $f \geq 0$ and $f \neq 0$.

(ii) We write $f \gg 0$ and say that f is *strongly positive* if $f(\omega) > 0$ for all $\omega \in K$.

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Definition

Let $E = C(K)$. The semigroup $(e^{tA})_{t \geq 0}$ is called *individually eventually strongly positive* if for each $f > 0$ there is a $t_0 \in [0, \infty)$ such that $e^{tA} f \gg 0$ for all $t \geq t_0$.

Theorem

If e^{tA} is compact for large t, then the following assertions are equivalent:

(i) $(e^{tA})_{t \geq 0}$ *is individually eventually strongly positive.*

(ii) $s(A)$ *is a simple and dominant eigenvalue of* A *and* $\ker(s(A) - A) = \langle u \rangle$ *for some* $u \gg 0$.

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A glimpse of the proof.

"(ii) $\Rightarrow$ (i)" Assertion (ii) implies that the spectral projection P corresponding to $s(A)$ is strongly positive and that $e^{tA}\to P$ as $t\to\infty$.

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"(ii) $\Rightarrow$ (i)" Assertion (ii) implies that the spectral projection P corresponding to $s(A)$ is strongly positive and that $e^{tA} \to P$ as $t \to \infty$.

"(i) $\Rightarrow$ (ii)" To see that $s(A)$ is dominant:

- Split off the peripheral spectrum.
- Show that the corresponding restriction of the semigroup is positive.
- Apply Perron-Frobenius theory of positive semigroups. □

Remark

(i) *Further characterizations involve the resolvent of A or the spectral projection corresponding to $s(A)$.*

(ii) *A generalization to arbitrary Banach lattices is possible under additional regularity assumptions on $(e^{tA})_{t \geq 0}$ and on the domain $D(A)$.*

Remark

(i) *Further characterizations involve the resolvent of A or the spectral projection corresponding to $s(A)$.*

(ii) *A generalization to arbitrary Banach lattices is possible under additional regularity assumptions on $(e^{tA})_{t\geq 0}$ and on the domain $D(A)$.*

(iii) *This generalization can be applied to study e.g. the semigroup generated by the bi-Laplacian on the disk in $\mathbb{R}^2$.*

For $x \in E$, let $d_+(x) := \operatorname{dist}(x, E_+)$ be the distance of x to the positive cone.

Definition

Suppose that $s(A) = 0$. The semigroup $(e^{tA})_{t \geq 0}$ is called...

(i) ...*uniformly asymptotically positive* if for each $\varepsilon > 0$ there is a $t_0 \in [0, \infty)$ such that $d_+(e^{tA}x) \leq \varepsilon \, ||x||$ for all $x \geq 0$ and for all $t \geq t_0$.

(ii) ...*individually asymptotically positive* if $\lim_{t \to \infty} d_+(e^{tA}x) = 0$ for all $x \geq 0$.

Theorem

Suppose that $s(A) = 0$ and that $(e^{tA})_{t \geq 0}$ is bounded and eventually compact. Then the following assertions are equivalent:

(i) $(e^{tA})_{t \geq 0}$ *is individually asymptotically positive.*

(ii) $(e^{tA})_{t \geq 0}$ *is uniformly asymptotically positive.*

(iii) $s(A)$ *is a dominant eigenvalue and the corresponding spectral projection* P *is positive.*

(iv) e^{tA} *converges (in operator norm) to a positive mapping as* $t \to \infty$.

Literature

For the finite dimensional case, see e.g.

[1] D. Noutsos, *On Perron-Frobenius property of matrices having some negative entries*, Linear Algebra Appl., 412 (2005), 132-153.

[2] D. Noutsos and M. Tsatsomeros, *Reachability and holdability of nonnegative states*, SIAM Journal on Matrix Analysis and Applications, 30 (2008), 700–712.

For the Dirichlet-to-Neumann operator which motivated this work, see

[3] D. Daners, *Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator*, Positivity, 18 (2014), 235-256.

For eventual positivity of the bi-Laplacian, see e.g.

[4] A. Ferrero, F. Gazzola, and H.-C. Grunau, *Decay and eventual local positivity for biharmonic parabolic equations*, Discrete Contin. Dyn. Syst., 21 (2008), 1129-1157.