Long term behaviour of positive operator semigroups

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Joint work with Moritz Gerlach (University of Potsdam)

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There is a sensible generalisation of this notion to the case where E is a Banach lattice.

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Simple generalisations:

- E is allowed to be a Banach lattice with order continuous norm.
- It suffices if $(T_t)_{t\geq 0}$ is bounded instead of contractive.

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Theorem (Pichór and Rudnicki, 2000)

Let $(T_t)_{t \geq 0}$ *be a Markov* C_0*-semigroup on* $E = L^1$ *with a fixed point* $f_0 \gg 0$.

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What about the irreducibility assumption?

- It suffices if K "interacts" with the entire semigroup.
- More precisely: It suffices that $Kf \neq 0$ for every fixed point $0 \neq f \geq 0$ of the semigroup (& that a weak technical assumption be fulfilled).

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$$V_t^{(0)} = T_t,$$
$$V_t^{(1)} = \int_0^t T_{t-s} K T_s \, \mathrm{d}s.$$

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S_t dominates $V_t^{(1)}$ which is a kernel operator for $t > 0$!

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Rest of the talk:
No time regularity is needed to prove convergence results.

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- stochastic continuity (on spaces of measures),
- even less continuity (for instance, liftings of semigroups to ultra products).

Don't prove theorems for each of those single cases!

Simply prove theorems without any time regularity.

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Example

For $S = (0,\infty)$ and $S = \mathbb{N}$, this yields the usual convergence as $t \to \infty$.

A convergence theorem without time regularity.

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Let $(T_t)_{t \in S}$ *be a positive, bounded semigroup representation on a Banach lattice with o.c. norm. Suppose that* $(T_t)_{t \in S}$ *has a fixed point* $f_0 \gg 0$ *and that* T_{t_0} *is a kernel operator for some* $t_0 \in S$.

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Then $(T_t)_{t \in S}$ is strongly convergent, provided that G is "good".

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Lemma

*Let $(T_t)_{t \in G}$ be a positive and bounded **group** representation on an **atomic** Banach lattice with o.c. norm. If G is "good" and if the representation has a fixed point $f_0 \gg 0$, then $T_t = \mathrm{id}$ for all $t \in G$.*

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- We thus obtain a group action of G on a finite subset of $\mathbb{N}$.

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$\Rightarrow$ The Lemma holds for every divisible group G.

A convergence theorem without time regularity.

Theorem (Gerlach and G., article in preparation)

Let $(G, +)$ be a commutative group and let $S \subseteq G$ be a subsemigroup such that $\langle S \rangle = G$.

Let $(T_t)_{t \in S}$ be a positive, bounded semigroup representation on a Banach lattice with o.c. norm. Suppose that the semigroup has a fixed point $f_0 \gg 0$ and that T_{t_0} is a kernel operator for some $t_0 \in S$.

Then $(T_t)_{t \in S}$ is strongly convergent, provided that G is "good".

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A similar result holds if T_{t_0} only dominates a kernel operator.

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Examples

(a) $\mathbb{R} = \langle [0, \infty) \rangle$ is divisible.

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- *The existence of **some** roots is not sufficient. We need roots of **every** order.*

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