Dilation Theorems on General Banach Spaces

Jochen Glück

Ulm University

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Joint work with Stephan Fackler (Ulm University)

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Simple consequences: T is contractive, J is isometric and $JQ \in \mathcal{L}(Y)$ is a projection onto $J(X)$.

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$\Rightarrow$ Useless!

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Given a Banach space X and a contraction $T \in \mathcal{L}(X)$, choose a class of Banach spaces $\mathcal{X}$ **with similar regularity as** X and ask:

Does T have a dilation within $\mathcal{X}$?

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Some applications:

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(b) The pointwise ergodic theorem is a consequence of the Akcoglu–Sucheston Dilation Theorem.

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(a) A structure theoretic part that works on general classes of Banach spaces.

(b) A geometric part where the properties of a concrete class of Banach spaces come into play.

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(c) The class of all uniformly convex Banach spaces, subject to a quantitative restraint on the uniform convexity.

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such that the following diagram commutes for every non-commutative polynomial p in n variables:

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Suppose that $\mathcal{T} \subseteq \mathcal{L}(X)$ has simultaneous dilations within $\mathcal{X}$. Then:

(a) *The multiplicative semigroup generated by $\mathcal{T}$ (i.e. the set $\{T_1 \cdot \ldots \cdot T_n : \; T_1, \dots, T_n \in \mathcal{T}\}$) has simultaneous dilations within $\mathcal{X}$.*

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(d) *Even the weak operator closure of $\mathcal{T}$ has simultaneous dilations within $\mathcal{X}$.*

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Example

All positive contractions on $L^p([0,1])$ (and more generally, on every L^p-space) have simultaneous dilations within the class of all L^p-spaces.