Eventual Positivity of Operator Semigroups

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Joint work with Daniel Daners (University of Sydney) and James B. Kennedy (University of Lisbon)

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Observation

Nobody has combined these two approaches, yet.

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(b) *individually eventually positive* if, for all $x \in E_+$, the inequality

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$$T^n f \to \int_0^1 f(x)\,dx \cdot \mathbb{1} \quad (n \to \infty).$$

If $f \geq 0$, then $T^n f \geq 0$ for all large n, **but**: this might happen very late if $\int_0^1 f(x)\,dx$ is small compared to $\|f\|_\infty$.

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(a) A (subtle) resolvent estimate.
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(c) Associate a *positive* operator S to the operator T by means of an ultra power argument. □

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(b) *The spectral bound $s(A)$ is a dominant spectral value of A; moreover, $\ker(s(A) - A)$ is spanned by a vector $v \gg_u 0$ and $\ker(s(A) - A')$ contains a strictly positive functional.*

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That's all certainly nice – but is it useful?

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Proof.

It follows from work of Grunau and Sweers [GS98] that $-\Delta^2$ (with the given boundary conditions) fulfils the spectral condition (b) in the above theorem. □

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Let Δ denote the Laplace operator with the above boundary conditions.

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Kreĭn–Rutman type argument $\Rightarrow$ condition (b) in the theorem holds. □

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- Can one obtain cyclicity results for the spectrum of eventually positive semigroups?
- Develop the perturbation theory of eventually positive semigroups until it reaches a satisfactory state.

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What do we not know, yet? – Open problems:

- Consider the line $s(A) + i\mathbb{R}$. Characterise eventual positivity of $(e^{tA})_{t \geq 0}$ if there exist essential spectral values and/or inifinitely many spectral values on this line.
- Can one obtain cyclicity results for the spectrum of eventually positive semigroups?
- Develop the perturbation theory of eventually positive semigroups until it reaches a satisfactory state.

Your turn!

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