Convergence Theorems for Positive Operator Semigroups

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Joint work with Moritz Gerlach (Universität Potsdam)

Linear evolution equations

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Example (The heat equation)

$E = L^1(\Omega)$ for a "nice" domain $\Omega \subseteq \mathbb{R}^d$ and A is the Laplace operator (with boundary conditions).

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(b) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$.

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Example

For each matrix $A \in \mathbb{C}^{n\times n}$ the operator family $(e^{tA})_{t\geq 0}$ is a operator semigroup on $\mathbb{C}^n$.

Long time behaviour

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(a) …**strongly convergent** if $\lim_{t\to\infty} T(t)u_0$ exists for every $u_0 \in E$.

(b) …**norm convergent** if $\lim_{t\to\infty} T(t)$ exists with respect to the operator norm.

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Interpretation: If $(T(t))_{t\geq 0}$ describes the solutions of

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then positivity of $(T(t))_{t\geq 0}$ means that positive initial values lead to positive solutions.

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Example: If $(T(t))_{t \geq 0}$ describes the solutions of the heat equation

$$\begin{cases} \dot{u}(t) = \Delta u(t), \\ u(0) = u_0, \end{cases}$$

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Example

The heat equation on $C(\overline{\Omega})$ (for bounded $\Omega \subseteq \mathbb{R}^d$ with Lipschitz boundary) with non-local Robin boundary conditions, compare [AKK, Thms A.1 and 6.3]

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Example

The solution of the heat equation on $L^1(\Omega)$ with Neumann boundary conditions is strongly convergent for bounded $\Omega \subseteq \mathbb{R}^d$ even if the boundary of Ω is "rough"; see [Are08].

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Example

Models from mathematical biology, e.g. the dynamics of immune states [DdGKT].

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Example

If a network flow on a metric graph is buffered in at least one vertex, then the flow is strongly convergent.

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