Universität Ulm

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Points: 18 + 5*

Exercise course in Functional Analysis: Problem Sheet 1

1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) Define $\ell^1 := \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \sum_{n=1}^\infty |x_n| < \infty\}$ and set $\|x\|_1 := \sum_{n=1}^\infty |x_n|$ for each $x \in \ell^1$. Show that $(\ell^1, \|\cdot\|_1)$ is a Banach space. (3)

(b) Define $c_0 := \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : x_n \to 0 \text{ as } n \to \infty\}$ and set $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$ for each $x \in c_0$. Show that $(c_0, \|\cdot\|_\infty)$ is a Banach space. (1)

2. Let V and W be normed vector spaces over the same scalar field and let $T : V \to W$ be linear. Show that the following assertions are equivalent: (4)

(i) If a sequence $(x_n)_{n\in\mathbb{N}}$ in V converges to a vector $x \in V$, then $(Tx_n)_{n\in\mathbb{N}}$ converges to Tx (i.e. T is continuous).

(i') If a sequence $(x_n)_{n\in\mathbb{N}}$ in V converges to 0, then $(Tx_n)_{n\in\mathbb{N}}$ converges to 0 (i.e. T is continuous in 0).

(ii) There exists a real number $c \geq 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in V$.

(iii) There exists a real number $\tilde{c} \geq 0$ such that $\|Tx\| \leq \tilde{c}$ for all x in the closed unit ball $\{v \in V : \|v\| \leq 1\}$.

Definition. Let V, W be normed vector spaces over the same scalar field.

(a) An *isomorphism* between V and W is a bijective linear mapping $\psi : V \to W$ such that both ψ and its inverse ψ^{-1} are continuous. The spaces V and W are called *isomorphic* if there exists an isomorphism between them.

(b) An *isometric isomorphism* between V and W is a bijective linear mapping $\psi : V \to W$ which is *isometric*, meaning that $\|\psi(x)\| = \|x\|$ for all $x \in V$. The spaces V and W are called *isometrically isomorphic* if there exists an isometric isomorphism between them.

3. Let V, W be normed spaces over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) Show that every isometric isomorphism $\psi : V \to W$ is also an isomorphism. (1)

(b) Assume that V and W are isomorphic. Show that V is a Banach space if and only if W is a Banach space. (1)

(c) For each $y \in \ell^1$ we define a mapping $Ty : c_0 \to \mathbb{K}$ by (3)

$$(Ty)(x) = \sum_{n=0}^\infty y_n x_n \quad \text{for all } x \in c_0.$$

Show that the above series converges, that Ty is an element of the dual space $(c_0)'$ and that the mapping $T : \ell^1 \to (c_0)'$, $y \mapsto Ty$ is an isometric isomorphism.

4. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We set $M := \{A = (a_{j,k})_{j,k\in\mathbb{N}} \subseteq \mathbb{K} : \sup_{j,k\in\mathbb{N}} |a_{j,k}| < \infty\}$. For each $A = (a_{j,k})_{j,k\in\mathbb{N}} \in M$ we define $\|A\|_\infty := \sup_{j,k} |a_{j,k}|$. It follows from the lecture that $(M, \|\cdot\|_\infty)$ is a Banach space.

For every $A \in M$ we define a mapping $\psi(A) : \ell^1 \to \ell^\infty$ by

$$\psi(A)x = (\sum_{k=1}^\infty a_{j,k} x_k)_{j\in\mathbb{N}} \quad \text{for } x \in \ell^1.$$

(a) Show that, for every $A \in M$, $\psi(A)$ is indeed a well-defined mapping from ℓ^1 to ℓ^∞; show also that $\psi(A)$ is linear and continuous, i.e. $\psi(A) \in \mathcal{L}(\ell^1; \ell^\infty)$. (2)

(b) Endow the space $\mathcal{L}(\ell^1;\ell^\infty)$ with the operator norm. Show that (3)

$$\psi : M \to \mathcal{L}(\ell^1;\ell^\infty), \quad A \mapsto \psi(A)$$

is an isometric isomorphism.

5. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and endow the space

$$c := \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \lim_{n\to\infty} x_n \text{ exists}\}$$

with the supremum norm given by $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$. It is not difficult to show that $(c, \|\cdot\|_\infty)$ is a Banach space.

(a) Show that the dual spaces of c_0 and c are isometrically isomorphic. (3*)

(b) Show that c_0 and c are isomorphic. (2*)

Fun fact: One can prove that c_0 and c are *not* isometrically isomorphic.