Universität Ulm

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Points: 22 + 5*

Exercise Course in Functional Analysis: Problem Sheet 2

6. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) Let V be a normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $U \subseteq V$ be a dense vector subspace. We endow U with the norm induced by V and consider U as a normed space in its own right. Define the mapping $T : V' \to U'$ by (2)

$$T(\varphi) = \varphi|_U \quad \text{for all } \varphi \in V';$$

here, $\varphi|_U$ denotes the restriction of the mapping φ to U. Prove that T is an isometric isomorphism.

(b) We endow that space (1)

$$c_{00} := \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \text{ all but finitely many } x_n \text{ are } 0\}$$

with the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$. Show that the dual spaces $(c_0)'$ and $(c_{00})'$ are isometrically isomorphic.

7. Let E be a Banach space over $\mathbb{C}$ and let $P \in \mathcal{L}(E)$ be a *projection*, meaning that $P^2 = P$.

(a) Prove that either $P = 0$ or $\|P\| \geq 1$. (1)

(b) Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > \|P\|$. Show that $(\lambda I - P)$ is invertible and compute $(\lambda I - P)^{-1}$. (2)
Hint: Neumann series!

(c) For which $\lambda \in \mathbb{C}$ is $\lambda I - P$ invertible? (3)

8. Let $\mathbb{K} = \mathbb{R}$ and let K be a compact metric space. Consider a sequence of functions $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ which is decreasing in the sense that $f_{n+1}(x) \leq f_n(x)$ for all $x \in K$ and all $n \in \mathbb{N}$. Let $f \in C(K)$ and assume that, for every $x \in K$, $f_n(x) \to f(x)$ as $n \to \infty$. Prove that $f_n \to f$ with respect to the $\|\cdot\|_\infty$-norm. (5*)

Remark: This assertion is called Dini's Theorem.

9. Let $\mathbb{K} = \mathbb{R}$ and let $\mathcal{P} \subseteq C([-1, 1])$ be the space of all polynomial functions on $[-1, 1]$ with real coefficients. Let $m \in C([-1, 1])$ be given by $m(x) = |x|$ for all $x \in [-1, 1]$. Prove, without using the Weierstraß approximation theorem, that m is contained in the closure of $\mathcal{P}$ (with respect to the $\|\cdot\|_\infty$-norm). (4)

Hint: Let $\varepsilon \in (0, 1)$ and let $f_\varepsilon \in C([-1, 1])$ be given by $f_\varepsilon(x) = \sqrt{\varepsilon + x^2}$. First show that f_ε is contained in the closure of $\mathcal{P}$.

Definition. Let K be a compact metric space.

(a) A mapping $T : C(K) \to C(K)$ is called an *algebra homomorphism* if T is linear and if $T(f \cdot g) = (Tf) \cdot (Tg)$ for all $f, g \in C(K)$.

(b) A mapping $T : C(K) \to C(K)$ is called a *unital algebra homomorphism* if T is an algebra homomorphism and if $T\mathbb{1}_K = \mathbb{1}_K$.

10. Let K be a compact metric space, let $\mathbb{K} = \mathbb{R}$.

(a) We write $f \geq 0$ for a function $f \in C(K)$ if $f(x) \geq 0$ for all $x \in K$. (1)
Let $T : C(K) \to C(K)$ be an algebra homomorphism. Prove that $Tf \geq 0$ whenever $f \geq 0$.

(b) Prove that every algebra homomorphism $T : C(K) \to C(K)$ is continuous. (2)

(c) Let $S, T : C(K) \to C(K)$ be unital algebra homomorphisms, let $h \in C(K)$ be an injective function and assume that $Sh = Th$. Prove that $S = T$. (2)

(d) Let $T : C(K) \to C(K)$ be a unital algebra homomorphism, let $h \in C(K)$ be an injective function and assume that the sequence $(T^n h)_{n \in \mathbb{N}}$ converges (with respect to the $\|\cdot\|_\infty$-norm). Show that the sequence $(T^n f)_{n \in \mathbb{N}}$ converges for every $f \in C(K)$. (4)