UNIVERSITÄT ULM

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Points: 23 + 13*

Exercise Course in Functional Analysis: Problem Sheet 3

11. (a) Let $\mathbb{K} = \mathbb{R}$ and let $K \subseteq \mathbb{R}^n$ be a non-empty compact set. By $\mathcal{P}(K)$ we denote the space of all polynomial functions (in real n variables) on K which have real coefficients. Prove that $\mathcal{P}(K)$ is dense in $C(K)$ (with respect to the $\|\cdot\|_\infty$-norm). (2)

(b) Let $\mathbb{K} = \mathbb{C}$ and denote the closed unit disk in $\mathbb{C}$ by $\overline{D} = \{z \in \mathbb{C} : \ |z| \le 1\}$. By $\mathcal{P}(\overline{D})$ we denote space of all polynomial functions (in one complex variable) on $\overline{D}$ which have complex coefficients. Prove that $\mathcal{P}(\overline{D})$ is *not* dense in $C(\overline{D})$. (2)

Hint: You may use the following result which follows, for instance, from the maximum principle in complex analysis:

For all $p \in \mathcal{P}(\overline{D})$ there exists a complex number $z_0 \in \partial\overline{D} = \{z \in \mathbb{C} : \ |z| = 1\}$ such that $\max_{z\in\overline{D}} |p(z)| = |p(z_0)|$.

(c) Let $\mathbb{K} = \mathbb{C}$, let K be a compact metric space and let $A \subseteq C(K)$ be a subalgebra of $C(K)$ (i.e. a vector subspace of $C(K)$ such that $fg \in A$ for all $f, g \in A$) which has the following properties: (3)

(a) $\mathbb{1}_K \in A$.

(b) A separates the points of K.

(c) We have $\overline{f} \in A$ for all $f \in A$.

Show that A is dense in $C(K)$. *(This is the complex version of the Stone-Weierstraß theorem!)*

(d) Let $\mathbb{K} = \mathbb{C}$ and let $\mathbb{T} := \partial\overline{D} = \{z \in \mathbb{C} : \ |z| = 1\}$ denote the complex unit circle. A function $f \in C(\mathbb{T})$ is called a *trigonometric polynomial* if there exist an integer $n \in \mathbb{N}_0$ and complex numbers $a_{-n}, \dots, a_n$ such that $f(z) = \sum_{k=-n}^{n} a_k z^k$ for all $z \in \mathbb{T}$. Let $\mathcal{T}(\mathbb{T})$ denote the set of all trigonometric polynomials in $C(\mathbb{T})$. (2)

Prove that $\mathcal{T}(\mathbb{T})$ is dense in $C(\mathbb{T})$.

Remark: The following three problems deal with compactness in normed spaces and, in particular, in Banach spaces. If you feel unsure about the concept of compactness, you can read the brief reminder on the second page of this Exercise Sheet; you can also solve the additional problems there.

12. Let V, W be normed vector spaces. A linear mapping $T : V \to W$ is called *compact* if the set $T(\overline{B_1(0)})$ is relatively compact in W (here, $\overline{B_1(0)}$ denotes the closed unit ball in V).

(a) Prove that every compact linear mapping $T : V \to W$ is continuous. (2)

(b) Let $P \in \mathcal{L}(V)$ be a compact projection. Prove that the range of P is finite dimensional. (3)

13. Let $K \subseteq \mathbb{R}^d$ be a non-empty, compact set and let $\theta \in (0, 1]$. Fix $x_0 \in K$ and endow the Hölder space $C^\theta(K)$ with the norm $\|f\|_{C^\theta(K)} = [f]_\theta + |f(x_0)|$ for all $f \in C^\theta(K)$. According to the lecture, $(C^\theta(K), \|\cdot\|_\theta)$ is a Banach space and we have $C^\theta(K) \subseteq C(K)$. (3)

Show that the canonical embedding

$$j : C^\theta(K) \hookrightarrow C(K), \qquad f \mapsto f$$

is compact.

14. Let $C \subseteq \ell^1$. Prove that the following assertions are equivalent: (6)

(i) C is relatively compact.

(ii) C is bounded and $\sup_{x\in C} \sum_{k=m}^{\infty} |x_k| \to 0$ as $m \to \infty$.

A brief reminder of compactness.

Definition. Let (M,d) be a metric space and let $C \subseteq M$.

(a) The subset C is called *bounded* if there exist an element $x \in M$ and a real number $r > 0$ such that $C \subseteq B_r(x)$ (here, $B_r(x) := \{y \in M : d(y,x) < r\}$ denotes the open ball with radius r in M).

(b) The subset C is called *complete* if the metric space $(C, d|_{C\times C})$ is complete.

(c) The subset C is called *totally bounded* (or *pre-compact*) if for every $\varepsilon > 0$ there exist finitely many numbers $x_1, \dots, x_n \in C$ such that $\bigcup_{k=1}^n B_\varepsilon(x_k) \supseteq C$.

(d) The subset C is called *compact* if every open covering of C admits a finite subcovering (more precisely, this means the following: whenever $\bigcup_{\lambda\in\Lambda} U_\lambda \supseteq C$ for a family $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets of M, then there exist finitely many indices $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $\bigcup_{k=1}^n U_{\lambda_k} \supseteq C$).

Let C be a subset of a metric space (M,d). We point out the following two observations:

(a) Compactness is an *intrinsic* property of C, i.e. we do not need any information about the surrounding space M in order to decide whether C is compact.

This seems a bit counter-intuitive at first glance since open subsets of M are used in the definition of compactness of C. However, a moment of reflection shows that we can rewrite the definition in a way that uses only intersections of open subsets of M with the set C; and those intersections are exactly the open sets in the metric space $(C, d|_{C\times C})$.

(b) Suppose that (M,d) is complete; then C is complete if and only if C is closed in M.

One can prove the following important characterisation of compact sets:

Theorem. *Let (M,d) be a metric space and let $C \subseteq M$. The following assertions are equivalent:*

(i) C is compact.

(ii) C is totally bounded and complete.

(iii) Every sequence in C has a convergent subsequence whose limit is contained in C.

15. Let C be a subset of a metric space (M,d).

(a) Show that the following assertions are equivalent: (2*)

(i) C is bounded.

(ii) For every $x \in M$ there exists a real number $r > 0$ such that $C \subseteq B_r(x)$.

(b) Prove that if C is totally bounded, then C is bounded. (2*)

(c) Show that the following assertions are equivalent: (2*)

(i) C is totally bounded.

(ii) For each $\varepsilon > 0$ there exists finitely many numbers $x_1, \dots, x_n \in M$ such that $\bigcup_{k=1}^n B_\varepsilon(x_k) \supseteq C$

(d) Prove that if C is totally bounded, then so is its closure $\overline{C}$. (2*)

(e) Show that the following assertions are equivalent: (5*)

(i) C is *relatively compact*, i.e. the closure of C is compact.

(ii) C is totally bounded and $\overline{C}$ is complete.

(iii) Every sequence in C has a subsequence which converges to an element of M.