Universität Ulm

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Points: 24 + 11*

Exercise Course in Functional Analysis: Problem Sheet 4

16. For all $f, g \in C([-1,1])$ we define $(f|g) = \int_{-1}^{1} f(x)\overline{g(x)}\,\mathrm{d}x$ (compare Example (6.8)(d) from the lecture). Prove that the pre-Hilbert space $\big(C([-1,1]), (\,\cdot\,|\,\cdot\,)\big)$ is not a Hilbert space. (4)

17. Let E be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *sesqui-linear form* on E is a mapping $a : E \times E \to \mathbb{K}$ which fulfils the following properties for all $x, y, z \in E$ and all $\lambda, \mu \in \mathbb{K}$:

$$a(\lambda x + \mu y, z) = \lambda a(x,z) + \mu a(y,z),$$
$$a(z, \lambda x + \mu y) = \overline{\lambda} a(z,x) + \overline{\mu} a(z,y).$$

If the scalar field $\mathbb{K}$ is real, a sesqui-linear form may also be called a *bilinear form*.

Let a and b be sesqui-linear forms on E.

(a) Let $\mathbb{K} = \mathbb{C}$. Prove that $4a(x,y) = \sum_{k=0}^{3} i^k a(x + i^k y, x + i^k y)$ for all $x, y \in E$. (2)
Remark: This equality is usually called the *polarization identity*.

(b) Let $\mathbb{K} = \mathbb{C}$. Prove that $a = b$ and only if $a(x,x) = b(x,x)$ for all $x \in E$. (1)

(c) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Prove that $2a(x,y) + 2a(y,x) = a(x+y, x+y) - a(x-y, x-y)$ for all $x, y \in E$. (1)

(d) Let $\mathbb{K} = \mathbb{R}$ and assume in addition that a and b are *symmetric*, meaning that $a(x,y) = a(y,x)$ and $b(x,y) = b(y,x)$ for all $x, y \in E$. Prove that $a = b$ if and only if $a(x,x) = b(x,x)$ for all $x, y \in E$. (1)

18. Let $\big(E, (\,\cdot\,|\,\cdot\,)\big)$ be a pre-Hilbert space.

(a) Let $x, y \in E$. Show that $x = y$ if and only if $(x|z) = (y|z)$ for all $z \in E$. (2)

(b) Let $T, S \in \mathcal{L}(E)$. Prove that $T = S$ if and only if $(Tx|y) = (Sx|y)$ for all $x, y \in E$. (1)

(c) Let $T, S \in \mathcal{L}(E)$ and assume that $\mathbb{K} = \mathbb{C}$. Prove that $T = S$ if and only if $(Tx, x) = (Sx, x)$ for all $x \in E$. (1)
Hint: Use the polarization identity.

(d) Give a concrete counterexample to show that the assertion of (c) is false in the case $\mathbb{K} = \mathbb{R}$. (3)

19. For each sequence $x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ we set $\|x\|_2 := \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \in [0, \infty]$. Moreover, we define

$$\ell^2 := \{x = (x_n)_{n\in\mathbb{N}} : \ \|x\|_2 < \infty\}.$$

(a) Prove that $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_2 \|y\|_2$ for all sequences $x = (x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$; here, we set $0 \cdot \infty := \infty \cdot 0 := 0$. (3)

(b) Show that the mapping $(\,\cdot\,|\,\cdot\,) : \ell^2 \times \ell^2 \to \mathbb{K}$ which is given by $(x|y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$ for all $x, y \in \ell^2$, is well-defined and a scalar product. (1)

(c) Note that the norm induced by the scalar product $(\,\cdot\,|\,\cdot\,)$ coincides with the mapping $\|\cdot\|_2 : \ell^2 \to [0, \infty)$. (2)
Show that $\big(\ell^2, (\,\cdot\,|\,\cdot\,)\big)$ is a Hilbert space.

(d) Write down an orthonormal basis for ℓ^2 (and prove that it really is an orthonormal basis). (1)

(e) Let $C \subseteq \ell^2$. Give a characterisation for relative compactness of C. (3*)

20. (a) Let $\big(E, (\cdot\,|\,\cdot)\big)$ be a pre-Hilbert space. Prove the so-called *parallelogram identity* (1)

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in E$.

(b) Let $(E, \|\cdot\|)$ be a normed vector space and assume that the norm fulfils the parallelogram identity. Show that there exists a scalar product $(\cdot\,|\,\cdot)$ on E which induces the norm $\|\cdot\|$. (8**)

Remark: This is the so-called *Jordan-von-Neumann theorem.*

(c) Guess what the double star in problem (b) means. (0*)