16. For all \( f, g \in C([-1, 1]) \) we define \( (f|g) = \int_{-1}^{1} f(x)g(x) \, dx \) (compare Example (6.8)(d) from the lecture). Prove that the pre-Hilbert space \( (C([-1, 1]), \langle \cdot | \cdot \rangle) \) is not a Hilbert space.

17. Let \( E \) be a vector space over \( K \in \{\mathbb{R}, \mathbb{C}\} \). A \textit{sesqui-linear form} on \( E \) is a mapping \( a : E \times E \to K \) which fulfils the following properties for all \( x, y, z \in E \) and all \( \lambda, \mu \in K \):
\[
\begin{align*}
    a(\lambda x + \mu y, z) &= \lambda a(x, z) + \mu a(y, z), \\
    a(z, \lambda x + \mu y) &= \overline{\lambda} a(z, x) + \mu a(z, y).
\end{align*}
\]

If the scalar field \( K \) is real, a sesqui-linear form may also be called a \textit{bilinear form}.

Let \( a \) and \( b \) be sesqui-linear forms on \( E \).

(a) Let \( K = \mathbb{C} \). Prove that \( 4a(x, y) = \sum_{k=0}^{3} i^k a(x + i^k y, x + i^k y) \) for all \( x, y \in E \).

Remark: This equality is usually called the \textit{polarization identity}.

(b) Let \( K = \mathbb{C} \). Prove that \( a = b \) if and only if \( a(x, x) = b(x, x) \) for all \( x \in E \).

(c) Let \( K \in \{\mathbb{R}, \mathbb{C}\} \). Prove that \( 2a(x, y) + 2a(y, x) = a(x + y, x + y) - a(x - y, x - y) \) for all \( x, y \in E \).

(d) Let \( K = \mathbb{R} \) and assume in addition that \( a \) and \( b \) are \textit{symmetric}, meaning that \( a(x, y) = a(y, x) \) and \( b(x, y) = b(y, x) \) for all \( x, y \in E \). Prove that \( a = b \) if and only if \( a(x, x) = b(x, x) \) for all \( x, y \in E \).

18. Let \( (E, \langle \cdot | \cdot \rangle) \) be a pre-Hilbert space.

(a) Let \( x, y \in E \). Show that \( x = y \) if and only if \( \langle x|z \rangle = \langle y|z \rangle \) for all \( z \in E \).

(b) Let \( T, S \in \mathcal{L}(E) \). Prove that \( T = S \) if and only if \( \langleTx|y\rangle = \langle Sx|y \rangle \) for all \( x, y \in E \).

(c) Let \( T, S \in \mathcal{L}(E) \) and assume that \( K = \mathbb{C} \). Prove that \( T = S \) if and only if \( \langleTx|x\rangle = \langle Sx|x \rangle \) for all \( x \in E \).

Hint: Use the polarization identity.

(d) Give a concrete counterexample to show that the assertion of (c) is false in the case \( K = \mathbb{R} \).

19. For each sequence \( x = (x_n)_{n \in \mathbb{N}} \subseteq K \) we set \( \|x\| := \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \in [0, \infty] \). Moreover, we define \( \ell^2 := \{x = (x_n)_{n \in \mathbb{N}} : \|x\|_2 < \infty\} \).

(a) Prove that \( \sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_2 \|y\|_2 \) for all sequences \( x = (x_n)_{n \in \mathbb{N}} \) and \( y = (y_n)_{n \in \mathbb{N}} \) in \( K \); here, we set \( 0 \cdot \infty := \infty \cdot 0 := 0 \).

(b) Show that the mapping \( \langle \cdot | \cdot \rangle : \ell^2 \times \ell^2 \to K \) which is given by \( \langle x|y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n} \) for all \( x, y \in \ell^2 \), is well-defined and a scalar product.

(c) Note that the norm induced by the scalar product \( \langle \cdot | \cdot \rangle \) coincides with the mapping \( \| \cdot \|_2 : \ell^2 \to [0, \infty) \).

Show that \( (\ell^2, \langle \cdot | \cdot \rangle) \) is a Hilbert space.

(d) Write down an orthonormal basis for \( \ell^2 \) (and prove that it really is an orthonormal basis).

(e) Let \( C \subseteq \ell^2 \). Give a characterisation for relative compactness of \( C \).
20. (a) Let $(E, (\cdot | \cdot))$ be a pre-Hilbert space. Prove the so-called parallelogram identity
\begin{equation}
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2
\end{equation}
for all $x, y \in E$.

(b) Let $(E, \|\cdot\|)$ be a normed vector space and assume that the norm fulfills the parallelogram identity. Show that there exists a scalar product $(\cdot | \cdot)$ on $E$ which induces the norm $\|\cdot\|$.  

Remark: This is the so-called Jordan-von-Neumann theorem.

(c) Guess what the double star in problem (b) means.