Universität Ulm

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Points: 26 + 10*

Exercise Course in Functional Analysis: Problem Sheet 5

21. Let $I = [-\pi, \pi] \subseteq \mathbb{R}$ be endowed with the Lebesgue measure and let $L^2(I)$ be endowed with the standard scalar product $(\cdot|\cdot)$.

(a) Prove that $C(I)$ is dense in the Hilbert space $L^2(I)$. (6*)

(b) Let $B \subseteq C(I)$ be dense with respect to the $\|\cdot\|_\infty$-norm. Prove that B is also dense in the Hilbert space $L^2(I)$. (1)

(c) Let $\mathbb{K} = \mathbb{C}$. For each $k \in \mathbb{Z}$ we define $e_k \in C(I)$ by $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ for all $x \in I$. Show that $(e_k)_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(I)$. (3)

(d) Let $f \in L^2(I)$ be given by $f(x) = x$ for $x \in I$. Compute $(f|e_k)$ for all $k \in \mathbb{Z}$. (2)

(e) Compute $\sum_{k=1}^\infty \frac{1}{k^2}$. (2)

22. Let $(H, (\cdot|\cdot))$ be a Hilbert space and let $A \in \mathcal{L}(H)$ be a linear operator which fulfils $(Ax|y) = (x|Ay)$ for all $x, y \in H$.

(a) Prove that $(Ax|x) \in \mathbb{R}$ for all $x \in H$. (1)

For the rest of this problem we assume that there exists a number $\varepsilon > 0$ such that $(Ax|x) \geq \varepsilon \|x\|^2$ for all $x \in H$.

(b) Define $(x|y)_A := (x|Ay)$ for all $x, y \in H$. Prove that $(\cdot|\cdot)_A$ is a scalar product on H and that $(H, (\cdot|\cdot)_A)$ is also a Hilbert space. (3)

(c) Fix $z \in H$ and define a mapping $\varphi_z : H \to \mathbb{K}$ by $\varphi_z(x) = (x|z)$ for all $x \in H$. Prove that φ_z is a continuous linear functional on the Hilbert space $(H, (\cdot|\cdot)_A)$. (1)

(d) Prove that A is bijective. (3)

Hint: The theorem of Riesz–Fréchet is of great help in order to prove surjectivity of A.

23. Give an elementary proof (i.e. a proof that does not use the theorem of Baire) of the uniform boundedness principle on the Banach space $\mathbb{K}^n$. (3)

24. (a) Let E be a Banach space and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E)$. Show that, if T_n converges to an operator $T \in \mathcal{L}(E)$ with respect to the operator norm as $n \to \infty$, then T_n converges also strongly to T. (1)

(b) Find an example of a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(\ell^2)$ with the following two properties: (3)

(i) T_n converges strongly to 0 as $n \to \infty$.

(ii) $\|T^n\| = 1$ for all $n \in \mathbb{N}$.

(c) Let E be a Banach space, let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E)$ and let $(c_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ be a null sequence with the following property: For every $x \in E$ there exists a number $d_x \geq 0$ such that $\|T_n x\| \leq d_x c_n$ for all $n \in \mathbb{N}$. (3)

Obviously, this implies that T_n converges strongly to 0 as $n \to \infty$. Prove that T_n even converges to 0 with respect to the operator norm as $n \to \infty$.

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25. Let E be a Banach space, let $F \subseteq E$ be a closed vector subspace and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E)$. We say that F is ... (4*)

(i) ... *individually eventually invariant under* $(T_n)_{n\in\mathbb{N}}$ if, for each $x \in F$, there exists an integer $n_0 \in \mathbb{N}$ such that $T_n x \in F$ for all $n \geq n_0$.

(ii) ... *uniformly eventually invariant under* $(T_n)_{n\in\mathbb{N}}$ if there exists an integer $n_0 \in \mathbb{N}$ such that $T_n F \subseteq F$ for all $n \geq n_0$.

Prove that F is uniformly eventually invariant under $(T_n)_{n\in\mathbb{N}}$ if and only if F is individually eventually invariant under F.