Universität Ulm

Deadline: Thursday, 30 November 2017

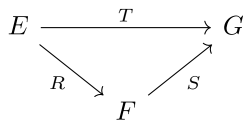
Prof. Dr. Wolfgang Arendt
Dr. Jochen Glück
Winter term 2017/18
Points: 20 + 17*

Exercise Course in Functional Analysis: Problem Sheet 6

26. Let $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ be a sequence such that $\lim_{N\to\infty} \sum_{k=1}^{N} x_n\overline{y_n}$ exists for every sequence $(x_n)_{n\in\mathbb{N}} \in \ell^2$. Prove that $(y_n)_{n\in\mathbb{N}} \in \ell^2$. (5)

Hint: Use the theorem of Banach–Steinhaus and the representation theorem of Riesz–Fréchet.

27. Let E, F be Banach spaces and let G be a normed space. Consider linear mappings $T : E \to G$, $R : E \to F$ and $S : F \to G$ such that the following diagram commutes: (4)



Show that, if T and S are continuous and S is injective, then R is continuous.

Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are called *equivalent* if there exist constants $c, d > 0$ such that $c\|x\|_1 \le \|x\|_2 \le d\|x\|_1$ for all $x \in V$. It is important to note that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if the identity mapping $\mathrm{id} : V \to V$ is an isomorphism between the normed spaces $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$.

28. Let V be a vector space over $\mathbb{K}$ and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two complete norms on V. Assume that there exists a constant $d > 0$ such that $\|x\|_2 \le d\|x\|_1$ for all $x \in V$.

(a) Fill in the details for Corollary (13.4) in the lecture, i.e. prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are actually equivalent. (2)

(b) Show that the conclusion of (a) fails in general if $\|\cdot\|_1$ is not assumed to be complete. (2)

(c) Show that the conclusion of (a) fails in general if $\|\cdot\|_2$ is not assumed to be complete. (2)

29. Let V be a vector space over $\mathbb{K}$.

(a) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two complete norms on V with the following property: (3)

$(*)$ *Whenever a sequence $(x_n) \subseteq V$ converges to a vector $a \in V$ with respect to $\|\cdot\|_1$ and to a vector $b \in V$ with respect to $\|\cdot\|_2$, then $a = b$.*

Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

(b) Assume now that $\dim V = \infty$ and let $\|\cdot\|_1$ be a complete norm on V. Show that there exists a complete norm $\|\cdot\|_2$ on V such that the property $(*)$ above fails. (2)

Please turn page.

Several students complained about a lack of difficult problems on the latest exercise sheets. Fortunately, such an issue is easily resolved...

30. Let $(E, \|\cdot\|)$ be a Banach space over $\mathbb{R}$ and let $K \subseteq E$ be a *closed cone*, i.e. a closed non-empty subset of E with the following properties:

(i) We have $\alpha x + \beta y \in K$ for all $x, y \in K$ and all scalars $\alpha, \beta \in [0, \infty)$.

(ii) We have $K \cap (-K) = \{0\}$, where $-K := \{-x : x \in K\}$.

Set $F := K - K := \{x - y : x, y \in K\}$; it is easy to see that F is a vector subspace of E.

(a) For each $z \in F$ we define $\|z\|_F := \inf\{\|x\| + \|y\| : \; x, y \in K \text{ and } z = x - y\}$. Prove that $\|\cdot\|_F$ is a complete norm on F. (4*)

(b) Assume that the cone K is *generating* in E, i.e. that $K - K = E$. Prove that there exists a number $c \geq 0$ with the following property: for every $z \in E$ there exist vectors $x, y \in K$ such that $z = x - y$ and $\|x\| + \|y\| \leq c\|z\|$. (2*)

For the rest of this problem we assume that the cone K is generating in E and that $\varphi : E \to \mathbb{R}$ is a linear mapping which is *positive*, meaning that $\varphi(x) \geq 0$ for all $x \in K$.

(c) Prove that there exists a number $d \geq 0$ such that $\varphi(x) \leq d\|x\|$ for all $x \in K$. (3*)

(d) Prove that φ is continuous. (1*)

In case that anybody is in need of a further challenge...

31. Let the scalar field be real and let $C_0(\mathbb{R})$ be the space of all scalar-valued continuous functions f on $\mathbb{R}$ which vanish at infinity (i.e. $\lim_{|x|\to\infty} f(x) = 0$). We endow this space with the $\|\cdot\|_\infty$-norm, which renders it a Banach space.

(a) A linear mapping $T : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ is called *positive* if $Tf \geq 0$ for all $f \geq 0$ (by $f \geq 0$ we mean that $f(x) \geq 0$ for all $x \in \mathbb{R}$). (4*)

Prove that every positive linear mapping $T : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ is continuous.

(b) Let $\Phi : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be an algebra homomorphism, i.e. a linear mapping which fulfils $T(fg) = (Tf)(Tg)$ for all $f, g \in C_0(\mathbb{R})$. Prove that T is continuous and that $\|T\| \leq 1$. (3*)