Universität Ulm

Deadline: Thursday, 07 December 2017

Prof. Dr. Wolfgang Arendt
Dr. Jochen Glück

Winter term 2017/18
Points: 20 + 16*

Exercise Course in Functional Analysis: Problem Sheet 7

32. (a) Consider the Banach space $\mathrm{C}([-1,1])$ (which we endow, as usual, with the $\|\cdot\|_\infty$-norm) and its closed vector subspace $F := \{f \in \mathrm{C}([-1,1]) : f|_{[-1,0]} = 0\}$. Prove that the quotient space $\mathrm{C}([-1,1])/F$ is isometrically isomorphic to the Banach space $\mathrm{C}([-1,0])$. (4)

(b) Consider the Banach space $c := \{(x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n \text{ exists}\}$ (endowed with the ∞-norm) and its closed vector subspace c_0. Find a concrete Banach space G which is isometrically isomorphic to the quotient space c/c_0. (2)

33. Let E be a Banach space and let $F, G \subseteq E$ be closed vector subspaces such that $E = F \oplus G$. Prove that the quotient space E/F is isomorphic to the Banach space G. (3)

34. Let M be a subset of a Banach space E. We say that M is *weakly bounded* if the set $\{\langle x', x\rangle : x \in M\}$ is bounded for every $x' \in E'$. (4)

Prove that M is weakly bounded if and only if M is bounded.

35. Let $\mathbb{K} = \mathbb{R}$, let K be a compact metric space and let $E = C(K)$. A linear functional $\psi \in E'$ is called *positive* if $\langle \psi, f\rangle \geq 0$ for all $f \geq 0$.

(a) Define $p : E \to \mathbb{R}$ by $p(f) := \|f^+\|_\infty$ for all $f \in E$ (where $f^+(x) := f(x) \vee 0$ for all $x \in K$). Show that p is sublinear. (2)

(b) Let $0 \leq f \in E$, let F denote the span of f and let $\varphi : F \to \mathbb{R}$ be given by $\varphi(\lambda f) = \lambda\|f\|$ for all $\lambda \in \mathbb{R}$. Prove that $\varphi(g) \leq p(g)$ for all $g \in F$. (2)

(c) Let $0 \leq f \in E$. Show that there exists a functional $\varphi \in E'$ which is positive and which fulfils $\|\varphi\| = 1$ and $\langle \varphi, f\rangle = \|f\|_\infty$. (3)

36. Let E be a real Banach space and let $E \subseteq E_+$ be a closed cone (cf. Problem 30 on Sheet 6).

(a) Define $p : E \to \mathbb{R}$ by $p(x) = \operatorname{dist}(x, -E_+)$ for each $x \in E$. Show that p is sublinear. (2*)

(b) Let $x \in E \setminus E_+$ and denote the span of x by F. We define $\varphi : F \to \mathbb{R}$ by $\varphi(\lambda x) = -\lambda \operatorname{dist}(x, E_+)$ for all $\lambda \in \mathbb{R}$. Prove that $\varphi(y) \leq p(y)$ for all $y \in F$. (2*)

(c) Let $x \in E \setminus E_+$. Show that there exists a positive functional $x' \in E'$ such that $\langle x', x\rangle < 0$. (2*)

(d) Show that a vector $x \in E$ fulfils $x \in E_+$ if and only if $\langle x', x\rangle \geq 0$ for all positive functionals $x' \in E'$. (1*)

(e) Let $x \in E$. Prove that there exists a positive functional $x' \in E'$ such that $\langle x', x\rangle \neq 0$. (2*)

(f) Let F also be a real Banach space and let $F_+ \subseteq F$ be a closed cone. A linear mapping $T : E \to F$ is called *positive* if $TE_+ \subseteq F_+$. (2*)

Assume that the cone E_+ is generating in E. Prove that every positive linear mapping $T : E \to F$ is continuous.

37. Let $\mathbb{K} = \mathbb{C}$. We endow the vector space $\mathrm{C}^1([0,1]) := \{f : [0,1] \to \mathbb{R} : f \text{ is continuously differentiable}\}$ with the norm $\|\cdot\|_{C^1}$ given by $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$ for all $f \in \mathrm{C}^1([0,1])$. It is not difficult to see that $(\mathrm{C}^1([0,1]), \|\cdot\|_{C^1})$ is a Banach space. Moreover, $\mathrm{C}^1([0,1])$ is an algebra with respect to pointwise multiplication, i.e. we have $fg \in \mathrm{C}^1([0,1])$ for all $f, g \in \mathrm{C}^1([0,1])$.

(a) Prove that every algebra homomorphism $\Phi : \mathrm{C}^1([0,1]) \to \mathrm{C}^1([0,1])$ is continuous. (3*)

(b) Give an example of an algebra homomorphism $\Phi : \mathrm{C}^1([0,1]) \to \mathrm{C}^1([0,1])$ which fulfils $\|\Phi\| > 1$. (2*)