Universität Ulm

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Points: 22 + 14*

Exercise Course in Functional Analysis: Problem Sheet 8

38. Consider the sequence $(f^{(n)})_{n\in\mathbb{N}} \subseteq c_0$ which is given by $f^{(n)} = (1, \dots, 1, 0, 0, \dots)$ for each $n \in \mathbb{N}$; here, the first n entries of the vector $f^{(n)}$ equal 1 and all further entries equal 0.

(a) Prove that $(f^{(n)})_{n\in\mathbb{N}}$ is not weakly convergent in c_0. (1)

(b) Let us consider the vectors $f^{(n)}$ as elements of ℓ^∞ now. Prove that the sequence $(f^{(n)})_{n\in\mathbb{N}}$ is weak*-convergent in ℓ^∞. (2)

39. (a) Let $f \in c_0$ and let $(f^{(n)})_{n\in\mathbb{N}} \subseteq c_0$ be a bounded sequence such that $f_k^{(n)} \to f_k$ as $n \to \infty$ for each $k \in \mathbb{N}$. Prove that the sequence $(f^{(n)})_{n\in\mathbb{N}}$ converges weakly to f. (2)

(b) Let $\mathbb{K} = \mathbb{R}$ and let $f, h \in c_0$. We write $f \le h$ iff $f_k \le h_k$ for all $k \in \mathbb{N}$. Moreover, we call the set $[f, h] := \{g \in c_0 : \ f \le g \le h\}$ the *order interval* between f and h (which is non-empty iff $f \le h$). (3)

Let $f \le h$. Prove that every sequence $(g^{(n)})_{n\in\mathbb{N}} \subseteq [f, h]$ has a weakly convergent subsequence.

Remark: This shows that order intervals in c_0 are *weakly sequentially compact.*

40. Let H be a pre-Hilbert space and let $(x_n)_{n\in\mathbb{N}} \subseteq H$ and $x \in H$. Show that the following assertions are equivalent: (3)

(i) $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to the norm on H.

(ii) $(x_n)_{n\in\mathbb{N}}$ converges weakly to x and $\limsup_{n\to\infty} \|x_n\| \le \|x\|$.

41. Let K be a compact metric space.

(a) Assume that K is infinite. Show that there exists a convergent sequence $(x_n)_{n\in\mathbb{N}}$ such that all elements x_n are pairwise distinct. (2)

(b) Assume that K is infinite. Construct a bounded sequence $(f_n)_{n\in\mathbb{N}}$ which does not have a weakly convergent subsequence. Conclude that $C(K)$ is not reflexive. (4)

(c) Prove that $C(K)$ is separable. (5)

Remark: In contrast to most results about $C(K)$-spaces which occur in this course, the separability of $C(K)$ relies heavily on the fact that K is a metric space and not merely a compact topological space!

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42. Let E be a separable Banach space. Show that there exists a compact metric space K and a closed vector subspace F of $C(K)$ such that E and F are isometrically isomorphic. (4*)

43. Let E be a real Banach space and let E_+ be a closed cone in E. The pair (E, E_+) is usually called an *ordered Banach space*. For two vectors $f, h \in E$ we write $f \le h$ iff $h - f \in E_+$. Moreover, we define the *order interval* $[f, h] := \{g \in E : \ f \le g \le h\}$ for all $f, h \in E$.

Assume throughout this exercise that the cone E_+ is generating.

(a) Let $f \in E$ and let $(g_n)_{n\in\mathbb{N}} \subseteq E_+$ be a sequence which converges to 0. Prove that $f \le 0$ if $f \le g_n$ for all $n \in \mathbb{N}$ and prove that $f = 0$ if $0 \le f \le g_n$ for all $n \in \mathbb{N}$. (1*)

(b) The cone E_+ is called *normal* if there exists a constant $C \ge 0$ such that $\|f\| \le C\|g\|$ for all vectors $f, g \in E$ which fulfil $0 \le f \le g$. Prove that the following assertions are equivalent: (5*)

- (i) The cone E_+ is normal.
- (ii) For each $g \in E_+$ the order interval $[0, g]$ is bounded.
- (iii) Every order interval in E is bounded.
- (iv) If two sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subseteq E_+$ fulfil $f_n \le g_n$ for all $n \in \mathbb{N}$ and if $(g_n)_{n\in\mathbb{N}}$ converges to 0, then $(f_n)_{n\in\mathbb{N}}$ converges to 0, too (compare (a)!).

(c) We define $E'_+ := \{x' \in E' : \ x' \text{ is positive}\}$. Prove that E'_+ is a closed cone in E'. (2*)
We call E'_+ the dual cone of E_+.

(d) Assume that the dual cone E'_+ is generating in E'. Prove that the cone E_+ is normal. (2*)