Universität Ulm

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Points: 20 + 13*

Exercise Course in Functional Analysis: Problem Sheet 9

44. Let E be a normed vector space and let $K, C \subseteq E$ be closed and convex subsets. Assume that K is compact and that $K \cap C = \emptyset$. Prove that there exist a number $\gamma \in \mathbb{R}$ and a functional $\varphi \in E'$ such that (5)

$$\mathrm{Re}\langle \varphi, x\rangle < \gamma < \mathrm{Re}\langle \varphi, y\rangle \qquad \forall x \in K \; \forall y \in C.$$

Hint: First prove that there exists a number $\delta > 0$ such that $\mathrm{dist}(y, K) \geq \delta$ for all $y \in C$.

45. Recall that $\ell^p := \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \; \|x\|_p := \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} < \infty\}$ for each $p \in [1, \infty)$. (5)
Let $1 \leq p_1 \leq p_2 \leq \infty$. Prove that

$$\ell^1 \subseteq \ell^{p_1} \subseteq \ell^{p_2} \subseteq \ell^\infty$$

and that the inclusion maps between these spaces are continuous.

Remark: Note that the inclusions above are converse to the inclusions for L^p-spaces over finite measure spaces!

46. (a) Let $p \in [1, \infty)$ and let $C \subseteq \ell^p$. Give a characterisation for relative compactness of C. (2)

(b) As usual, let c denote the space of all convergent sequences, Let $C \subseteq c$. Give a characterisation for relative compactness of C. (3)

47. (a) Prove that every pre-Hilbert space is uniformly convex. (3)

(b) Prove that the spaces c_0 and c are not uniformly convex. (2)

48. In this exercise we give a purely geometric proof of the Hahn–Banach separation theorem. Do not use the Hahn–Banach extension theorem or any of the separations theorems in the proofs!

Let $\mathbb{K} = \mathbb{R}$. Let B be a non-empty open convex subset of a normed space E and let $a \in E \setminus B$.

(a) For all $x, y \in E$ denote the line segment between x and y by $[x, y] := \{x + t(y - x) : \; t \in [0, 1]\}$ (note that this has nothing to do with order intervals in ordered Banach spaces, for which we used the same notation). Moreover, for each $S \subseteq E$ we denote the convex hull of S by $\mathrm{conv}\, S$. Prove that we have $\mathrm{conv}(C \cup \{x\}) = \bigcup_{c\in C}[c, x]$ for each convex set $\emptyset \neq C \subseteq E$ and each $x \in E$. (2*)

(b) Show that there exists a maximal set H (with respect to set inclusion) among all convex subsets of E that contain a and do not intersect B. Show moreover that H is closed. (3*)

(c) Prove that $E \setminus H$ is convex, too. (4*)

(d) Fix an arbitrary vector $h \in \partial H$. Prove that $F := \partial H - h$ is closed vector subspace of E and that E/F has dimension 1. Conclude that there exists a functional $\varphi \in E'$ whose kernel coincides with F. (3*)

(e) Show that there exists a functional $\varphi \in E'$ such that $\langle \varphi, b\rangle < \langle \varphi, a\rangle$ for all $b \in B$. (1*)