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Points: 21 + 9*

# Exercise Course in Functional Analysis: Problem Sheet 10

**49.** Let $E$ be a Banach space and let $T \in \mathcal{L}(E)$.

(a) Assume that $T$ is bijective. Prove that the adjoint $T' \in \mathcal{L}(E')$ is also bijective and that we have $(T')^{-1} = (T^{-1})'$. (2)

(b) Assume that $T'$ is bijective. Prove that $T$ is also bijective. (4)
*Hint: First show that $T$ is injective and that the range of $T$ is dense in $E$. Then use the bi-adjoint of $T$.*

**50.** Let $(X, \tau)$ be a topological space and let $C \subseteq X$. Show that the following assertions are equivalent: (4)

(i) $X$ is closed.

(ii) Whenever a net $(x_j)_{j\in J}$ in $C$ converges to an element $x \in X$, then $x \in C$.

**51.** Let $(X, \tau)$ be a topological space. Show that the following assertions are equivalent: (4)

(i) $(X, \tau)$ is a Hausdorff space.

(ii) For any two distinct points $x, y \in X$ there exist disjoint open sets $A, B \subseteq X$ such that $x \in A$ and $y \in B$.

**52.** Let $X, Y$ be non-empty sets and let $S \subseteq X$. Let $(x_i)_{i\in I}$ be a net in $X$ and let $f : X \to Y$ be an arbitrary mapping.

(a) Show that if $(x_i)_{i\in I}$ is a universal net in $X$, then $(f(x_i))_{i\in I}$ is a universal net in $Y$. (1)

(b) Assume that $x_i \in S$ for all $i \in I$. Prove that the net $(x_i)_{i\in I}$ is universal in $X$ if and only if it is universal in $S$. (2)

(c) Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $X$ which is a universal net. Prove that this sequence is eventually constant. (2)

(d) Give an example of two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with the following properties: (2)

(1) $(a_n)_{n\in\mathbb{N}}$ is a subnet of $(b_n)_{n\in\mathbb{N}}$.
(2) $(a_n)_{n\in\mathbb{N}}$ is not a subsequence of $(b_n)_{n\in\mathbb{N}}$.

**53.** This exercise has been removed since it turned out that the exercise did not really make sense.

Please turn page.

**54.** Let $Y$ be a non-empty set. A *filter* on $Y$ is a collection of sets $\mathcal{F} \subseteq \mathcal{P}(Y)$ with the following properties:

(1) Every set $A \in \mathcal{F}$ is non-empty.

(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(3) If $A \subseteq B \subseteq Y$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

An *ultra filter* on $Y$ is a filter on $Y$ which is maximal (with respect to inclusion) among all filters on $Y$.

(a) Prove that every filter on $Y$ is contained in an ultra filter on $Y$. (2*)

(b) Let $I$ be a directed set. Prove that there exists an ultra filter on $I$ which contains all sets of the type $\{i \in I : \; i \geq i_0\}$ (where $i_0$ runs through $I$). (3*)

(c) Let $(x_i)_{i \in I}$ be a net in a non-empty set $X$. Prove that $(x_i)_{i \in I}$ has a subnet which is a universal net. (4*)