Exercise Course in Functional Analysis: Problem Sheet 11

55. Let $E$ be a Banach space and let $K \subseteq E$ be convex and weakly compact. Let $\varphi : K \to \mathbb{R}$ be convex and lower semicontinuous. In this exercise we give an alternative proof of the fact that $\varphi$ has a minimum.

(a) Define $S_\lambda = \{ x \in K : \varphi(x) \leq \lambda \}$ for each $\lambda \in \mathbb{R}$. Show that $S_\lambda$ is norm closed and convex for each $\lambda \in \mathbb{R}$. 

(b) Show that $S_\lambda$ is weakly compact for each $\lambda \in \mathbb{R}$. 

(c) Show that $\bigcap_{\lambda \in \varphi(K)} S_\lambda$ is non-empty and conclude that $\varphi$ has a minimum.

56. Let $E$ and $F$ be Banach spaces and let $T \in \mathcal{L}(E)$. Let $\tau_E$ and $\tau_F$ denote the weak topologies on $E$ and $F$, respectively. Prove that $T : (E, \tau_E) \to (F, \tau_F)$ is continuous.

57. Let $E$ be a normed vector space and let $K \subseteq E$ be a convex set.

(a) Let $x \in K$ be arbitrary and let $y \in K$ be an interior point of $K$. Show that $(1 - \lambda)x + \lambda y$ is an interior point of $K$ for every $\lambda \in (0, 1]$.

(b) Assume that $K$ is closed and has non-empty interior. Prove that $K$ coincides with the closure of its interior.

For the rest of this exercise we assume that the scalar field is real.

(c) Let $x \in K$. A functional $\varphi \in E' \setminus \{0\}$ is said to support $K$ at $x$ if $\langle \varphi, y \rangle \geq \langle \varphi, x \rangle$ for all $y \in K$. Assume that $K$ is closed and has non-empty interior, and let $x \in \partial K$. Show that there exists a functional $\varphi \in E' \setminus \{0\}$ which supports $K$ at $x$.

(d) Now, endow $[0, 1]$ with the Lebesgue measure and set $E = L^1([0, 1])$. Let $K := \{ f \in E : f \geq 0 \text{ almost everywhere} \}$. Show that $K$ is a closed convex set with empty interior. Show moreover that there does not exist a functional $\varphi \in E' \setminus \{0\}$ which supports $K$ at $1_{[0,1]}$. 

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