Universität Ulm

Deadline: Thursday, 18 January 2018

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Winter term 2017/18
Points: 23

Exercise Course in Functional Analysis: Problem Sheet 11

55. Let E be a Banach space and let $K \subseteq E$ be convex and weakly compact. Let $\varphi : K \to \mathbb{R}$ be convex and lower semicontinuous. In this exercise we give an alternative proof of the fact that φ has a minimum.

(a) Define $S_\lambda = \{x \in K : \varphi(x) \leq \lambda\}$ for each $\lambda \in \mathbb{R}$. Show that S_λ is norm closed and convex for each $\lambda \in \mathbb{R}$. (2)

(b) Show that S_λ is weakly compact for each $\lambda \in \mathbb{R}$. (3)

(c) Show that $\bigcap_{\lambda \in \varphi(K)} S_\lambda$ is non-empty and conclude that φ has a minimum. (4)

56. Let E and F be Banach spaces and let $T \in \mathcal{L}(E)$. Let τ_E and τ_F denote the weak topologies on E and F, respectively. Proof that $T : (E, \tau_E) \to (F, \tau_F)$ is continuous. (4)

57. Let E be a normed vector space and let $K \subseteq E$ be a convex set.

(a) Let $x \in K$ be arbitrary and let $y \in K$ be an interior point of K. Show that $(1-\lambda)x + \lambda y$ is an interior point of K for every $\lambda \in (0, 1]$. (2)

(b) Assume that K is closed and has non-empty interior. Prove that K coincides with the closure of its interior. (2)

For the rest of this exercise we assume that the scalar field is real.

(c) Let $x \in K$. A functional $\varphi \in E' \setminus \{0\}$ is said to *support* K *at* x if $\langle \varphi, y \rangle \geq \langle \varphi, x \rangle$ for all $y \in K$. Assume that K is closed and has non-empty interior, and let $x \in \partial K$. Show that there exists a functional $\varphi \in E' \setminus \{0\}$ which supports K at x. (2)

(d) Now, endow $[0, 1]$ with the Lebesgue measure and set $E = L^1([0, 1])$. Let $K := \{f \in E : f \geq 0 \text{ almost everywhere}\}$.

Show that K is a closed convex set with empty interior. Show moreover that there does not exist a functional $\varphi \in E' \setminus \{0\}$ which supports K at $\mathbb{1}_{[0,1]}$. (4)