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Points: 18

Exercise Course in Functional Analysis: Problem Sheet 12

58. Let E be a Banach space over $\mathbb{K}$, let $T \in \mathcal{L}(E)$ and let $\lambda \in \rho(T)$. Prove that $\|R(\lambda, T)\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}$. (2)

Hint: Use Proposition (31.1).

59. Let E be a Banach space over $\mathbb{K}$ and let $T \in \mathcal{L}(E)$. Let $\lambda \in \rho(T)$ and $\mu \in \mathbb{K}$.

(a) Prove that the following assertions are equivalent: (2)

(i) μ is a spectral value of T.

(ii) $\frac{1}{\lambda - \mu}$ is a spectral value of $R(\lambda, T)$.

Remark: The above equivalence is called the *spectral mapping theorem for the resolvent*; it can be written in short form as $(\lambda - \sigma(T))^{-1} = \sigma(R(\lambda, T))$.

(b) Let $x \in E \setminus \{0\}$. Prove that the following assertions are equivalent: (2)

(i) μ is an eigenvalue of T and x is an associated eigenvector.

(ii) $\frac{1}{\mu - \lambda}$ is an eigenvalue of $R(\lambda, T)$ and x is an associated eigenvector.

Remark: The above equivalence implies that $(\lambda - \sigma_{\mathrm{p}}(T))^{-1} = \sigma_{\mathrm{p}}(R(\lambda, T))$; this equality is called the *spectral mapping theorem for the point spectrum and the resolvent.*

60. Let $p \in [1, \infty]$, let the scalar field be complex and set $\ell^p := \ell^p(\mathbb{N})$.

(a) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$ and define $T \in \mathcal{L}(\ell^p)$ by (4)

$$T(x_n)_{n\in\mathbb{N}} = (\alpha_n x_n)_{n\in\mathbb{N}}$$

for all $(x_n)_{n\in\mathbb{N}} \in \ell^p$. Compute the operator norm of T as well as its point spectrum, its approximate point spectrum and its spectrum.

(b) Define $L, R \in \mathcal{L}(\ell^p)$ by (8)

$$L(x_n)_{n\in\mathbb{N}} = (x_2, x_3, \dots) \qquad \text{and} \qquad R(x_n)_{n\in\mathbb{N}} = (0, x_1, x_2, \dots)$$

for all $(x_n)_{n\in\mathbb{N}} \in \ell^p$. The operator L is called the *left shift operator* and R is called the *right shift operator*. Note that $LR = \mathrm{id}$, but $RL \neq \mathrm{id}$.

Compute the operator norms of L and R as well as their point spectra, their approximate point spectra and their spectra.

Hint: At some point it might be helpful to compute the adjoint operators L' and R'.

Remark: The $4 + 8$ points for this exercise do not mean that the exercise is particularly difficult, but simply that there are 12 different objects which you are asked to compute (three norms, three point spectra, three approximate point spectra and three spectra). You can "partially vote" for this exercise by simply filling in the numbers of objects you computed in the voting list.