Exercise Course in Functional Analysis: Problem Sheet 13

61. Let $E$ be a Banach space and let $F \subseteq E$ be a closed subspace. The set

$$F^\perp := \{ x' \in E' : \langle x', x \rangle = 0 \text{ for all } x \in F \} \subseteq E'$$

is called the annihilator of $F$ in $E'$.

(a) Let $q : E \rightarrow E/F$ denote the quotient mapping and define $J : (E/F)' \rightarrow E'$ by $J(\varphi) = \varphi \circ q$ for each $\varphi \in (E/F)'$. Prove that $J$ is a Banach space isomorphism from $(E/F)'$ to $F^\perp$.

(b) Let $T \in \mathcal{L}(E)$. Prove that $(TE)^\perp = \ker T'$.

62. Let $E = C([0,1])$ and let $T \in \mathcal{L}(E)$ be given by

$$(Tf)(x) = \int_0^x f(y) \, dy \quad \forall x \in [0,1]$$

for each $f \in E$. One readily checks that $T \in \mathcal{L}(E)$.

(a) Prove that $T$ is compact.

(b) Prove that $\sigma_p(T) = \emptyset$ and that $\sigma(T) = \{0\}$.

63. Let $p \in [1,\infty)$. Similarly as in the Exercises 14 and 19(e) one can prove that a set $C \subseteq \ell^p$ is relatively compact in $\ell^p$ if and only if $C$ is bounded and fulfills

$$\sup_{x \in C} \sum_{k=m}^{\infty} |x_k|^p \to 0 \quad \text{as} \quad m \to \infty.$$ 

Let $(\alpha_n)_{n \in \mathbb{N}} \subseteq \mathbb{K}$ be bounded and let $T \in \mathcal{L}(\ell^p)$ be the multiplication operator given by $Tx = (\alpha_n x_n)_{n \in \mathbb{N}}$ for each $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$.

Show that $T$ is compact if and only if $\alpha_n \to 0$ as $n \to \infty$.

64. In this exercise we discuss a generalisation of the results at the end of Section 33 in the lecture. Let $p \in [1,\infty)$, let $(\Omega, \Sigma, \mu)$ be a finite measure space and set $L^p := L^p(\Omega, \Sigma, \mu)$ and $L^\infty := L^\infty(\Omega, \Sigma, \mu)$.

Let $T \in \mathcal{L}(L^p)$ and assume that $T L^p \subseteq L^\infty$.

(a) Prove that $T \in \mathcal{L}(L^p, L^\infty)$.

Hint: Use the closed graph theorem.

(b) Show that $T$ is a compact operator from $L^p$ to $L^p$ in case that $p \in (1,\infty)$ and show that $T^2$ is a compact operator from $L^p$ to $L^p$ in case that $p = 1$.

Hint: Use may use without proof that a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L^\infty$ which is weakly convergent in $L^\infty$ is norm convergent in $L^p$. 

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