Universität Ulm

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Points: 17 + 7*

Exercise Course in Functional Analysis: Problem Sheet 13

61. Let E be a Banach space and let $F \subseteq E$ be a closed subspace. The set

$$F^\perp := \{x' \in E' : \langle x', x \rangle = 0 \text{ for all } x \in F\} \subseteq E'$$

is called the *annihilator* of F in E'.

(a) Let $q : E \to E/F$ denote the quotient mapping and define $J : (E/F)' \to E'$ by $J(\varphi) = \varphi \circ q$ for each $\varphi \in (E/F)'$. Prove that J is a Banach space isomorphism from $(E/F)'$ to $F^\perp$. (5)

(b) Let $T \in \mathcal{L}(E)$. Prove that $(TE)^\perp = \ker T'$. (2)

62. Let $E = C([0, 1])$ and let $T \in \mathcal{L}(E)$ be given by

$$(Tf)(x) = \int_0^x f(y)\,\mathrm{d}y \qquad \forall x \in [0, 1]$$

for each $f \in E$. One readily checks that $T \in \mathcal{L}(E)$.

(a) Prove that T is compact. (3)

(b) Prove that $\sigma_{\mathrm{p}}(T) = \emptyset$ and that $\sigma(T) = \{0\}$. (3)

63. Let $p \in [1, \infty)$. Similarly as in the Exercises 14 and 19(e) one can prove that a set $C \subseteq \ell^p$ is relatively compact in ℓ^p if and only if C is bounded and fulfils (4)

$$\sup_{x \in C} \sum_{k=m}^{\infty} |x_k|^p \to 0 \quad \text{as} \quad m \to \infty.$$

Let $(\alpha_n)_{n \in \mathbb{N}} \subseteq \mathbb{K}$ be bounded and let $T \in \mathcal{L}(\ell^p)$ be the multiplication operator given by $Tx = (\alpha_n x_n)_{n \in \mathbb{N}}$ for each $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$.

Show that T is compact if and only if $\alpha_n \to 0$ as $n \to \infty$.

64. In this exercise we discuss a generalisation of the results at the end of Section 33 in the lecture. Let $p \in [1, \infty)$, let (Ω, Σ, μ) be a finite measure space and set $L^p := L^p(\Omega, \Sigma, \mu)$ and $L^\infty := L^\infty(\Omega, \Sigma, \mu)$. Let $T \in \mathcal{L}(L^p)$ and assume that $TL^p \subseteq L^\infty$. (2*)

(a) Prove that $T \in \mathcal{L}(L^p, L^\infty)$.
Hint: Use the closed graph theorem.

(b) Show that T is a compact operator from L^p to L^p in case that $p \in (1, \infty)$ and show that T^2 is a compact operator from L^p to L^p in case that $p = 1$. (5*)
Hint: Use may use without proof that a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L^\infty$ which is weakly convergent in L^∞ is norm convergent in L^p.