Universität Ulm

Deadline: Thursday, 8 February 2018

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Points: 20

Exercise Course in Functional Analysis: Problem Sheet 14

65. Let $\mathbb{K} = \mathbb{C}$. For every $\lambda \in \ell^\infty$ we define the operator $D_\lambda \in \mathcal{L}(\ell^2)$ by $D_\lambda x = \lambda x = (\lambda_n x_n)_{n \in \mathbb{N}}$ for each $x \in \ell^2$. (4)

Let H be an infinite dimensional separable Hilbert space and let $T \in \mathcal{L}(H)$. Prove that the following assertions are equivalent:

(i) T is normal and compact.

(ii) There exist a sequence $\lambda \in c_0$ and a unitary operator $U : \ell^2 \to H$ such that $T = U D_\lambda U^{-1}$.

66. Let H be a pre-Hilbert space and let $x, y \in H$. Show that the following assertions are equivalent: (3)

(i) $x \perp y$.

(ii) $\|x + \alpha y\| \geq \|x\|$ for each $\alpha \in \mathbb{K}$.

67. Let $\mathbb{K} = \mathbb{C}$, let H be a Hilbert space and let $T \in \mathcal{L}(E)$ be a contractive operator (i.e. an operator which fulfils $\|T\| \leq 1$). (4)

Let $\lambda, \mu \in \mathbb{C}$ be two distinct eigenvalues of T of modulus 1 and let $x, y \in H$ be corresponding eigenvectors, respectively. Show that $x \perp y$.

Hint: Apply the rescaled resolvent $(|\nu| - 1)R(\nu, T)$ to the vector $x + \alpha y$, where $\alpha \in \mathbb{C}$ is arbitrary and $\nu \in \rho(T)$ are certain appropriately chosen numbers.

68. Define a mapping $T : L^2([0,1]) \to L^2([0,1])$ by means of the formula

$$(Tf)(x) = \int_1^x \int_0^y f(z) \, \mathrm{d}z \, \mathrm{d}y \qquad (x \in [0,1])$$

for all $f \in L^2([0,1])$.

(a) Show that T is well-defined and a bounded linear operator on $L^2([0,1])$. (2)

(b) Show that T is self-adjoint. (4)
Hint: First proof that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in C([0,1])$.

(c) Show that T is compact. (3)