Universität Ulm

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Points: 20*

Exercise Course in Functional Analysis: Problem Sheet 15

69. You are hiking in a two-dimensional compact and convex hiking area K. Right in front of you, there is a map of the area. By a map we mean a tuple (M, m) where M is a subset of K and $m : K \to M$ is a continuous bijection. Of course you know how to read a map: every point $y \in M$ on the map corresponds to the point $m^{-1}(y)$ in the hiking area (some people might find it difficult to use the map if m is note affine, but you are a mathematician, so you're fine with an m which is merely continuous and bijective). (2*)

A *"You are here"-point* on the map is a point $p \in M$ with the following property: if you step upon the map and stand right upon the point p, then the position in K which is indicated by your position on the map coincides with your real position in K.

Prove that every map contains a "You are here"-point. Prove moreover that this remains true if the hiking area K is not two-dimensional but, more generally, a compact convex subset of a Banach space E.

70. Give an example of a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ which does not have a non-trivial invariant subspace. Why doesn't this contradict Lomonosov's theorem? (2*)

71. Give a counterexample which show's that Brouwer's fixed point theorem does not in general hold on compact subsets $K \subseteq \mathbb{R}^n$ which are merely connected, but not convex. (2*)

72. Let E be a Banach space.

(a) Let $\emptyset \neq K \subseteq E'$ be convex and compact with respect to the weak*-topology. Let $T \in \mathcal{L}(E)$ and fix $z' \in E'$. We define a mapping $\varphi : E' \to E'$ by $\varphi(x') = T'x' + z'$ for all $x' \in E'$. (4*)
Show that φ has a fixed point in K in case that $\varphi(K) \subseteq K$.
Hint: Choose an arbitrary point $x' \in K$ and consider a weak*-convergent subnet of the sequence $(\frac{1}{n} \sum_{k=0}^{n-1} \varphi^k(x'))_{n \in \mathbb{N}}$.

(b) Let $T \in \mathcal{L}(E)$ be power pounded, meaning that $M := \sup_{n \in \mathbb{N}} \|T^n\| < \infty$. (5*)
Assume that 1 is an eigenvalue of T. Prove that 1 is also an eigenvalue of the dual operator T'.
Hint: The set $\hat{K} := \{x' \in E' : \|(T')^n x'\| \leq 1 \text{ for all } n \in \mathbb{N}_0\}$ is quite useful.

(c) The following generalisation of (a) is a version of the *Kaktuani–Markov fixed point theorem.* (5*)
Let I be an arbitrary non-empty set and assume that, for each $i \in I$, a mapping $\varphi_i : E' \to E'$ is given by $\varphi_i(x') = T_i' x' + z_i'$ for all $x' \in E'$, where $T_i \in \mathcal{L}(E)$ and $z_i' \in E'$.
Let $\emptyset \neq K \subseteq E'$ be convex and compact with respect to the weak*-topology. Assume that $\varphi_i(K) \subseteq K$ for each $i \in K$ and that all the mappings φ_i commute on K (i.e. that $\varphi_i(\varphi_j(x')) = \varphi_j(\varphi_i(x'))$ for all $i, j \in I$ and all $x' \in K$.
Prove that the mappings φ_i $(i \in I)$ have a common fixed point in K, i.e. that there exists a vector $x_0' \in K$ such that $\varphi_i(x_0') = x_0'$ for all $i \in I$.
Hint: Zorn's lemma.