Universität Ulm

Submission: Wednesday, 03.05.2017

Prof. Dr. Wolfgang Arendt
Henrik Kreidler
Sommersemester 2017
Points total: 16 + 4*

# Evolutionsgleichungen: Exercise Sheet 2

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

**1.** Let $X$ be a Banach space and $A$ an operator on $X$ with domain $D(A)$. Equipped with the *graph norm* $\|\cdot\|$ defined by $\|x\|_A := \|x\| + \|Ax\|$ for $x \in D(A)$ the domain $D(A)$ is a normed space.

(i) Show that $A$ is a closed operator if and only if $(D(A), \|\cdot\|_A)$ is a Banach space. (2)

(ii) The operator $A$ is called *closable* if there is a closed operator $B$ on $X$ such that $A$ is a *restriction* of $B$ (in symbols: $A \subseteq B$), i.e., $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$. Show that if $A$ is closable, then there is a smallest closed operator $\overline{A}$ (called the *closure of* $A$) with $A \subseteq \overline{A}$. (2)

**2.** Determine the generator of the *diagonal semigroup* $M_q$ on $\mathrm{C}_0(\mathbb{R})$ of exercise sheet 1, exercise 4. (4)

**3.** Consider a $C_0$-semigroup $T$ on a Banach space $X$ with generator $(A, D(A))$. Show that the following semigroups $S$ are strongly continuous and determine their generators.

(i) $S(t) := e^{\alpha t} T(\beta t)$ for all $t \geq 0$ where $\alpha \in \mathbb{C}$ and $\beta > 0$ are fixed parameters. (1)

(ii) $S(t) := V^{-1} T(t) V$ for all $t \geq 0$ where $V : Y \longrightarrow X$ is an isomorphism from a Banach space $Y$ to $X$. (1)

(iii) $S(t) := T(t)|_Z$ for all $t \geq 0$ where $Z \subseteq X$ is a closed subspace of $X$ with $T(t)Z \subseteq Z$ for all $t \geq 0$. (1)

**4.** Given two functions $f, g \in \mathrm{L}^2(\mathbb{R})$ we define their *convolution* $f * g$ by

$$f * g(x) := \int_{\mathbb{R}} f(x-y) g(y) \, \mathrm{d}y \text{ for } x \in \mathbb{R}.$$

Moreover, we denote the *Fourier transformation* by $\mathscr{F} \in \mathscr{L}(\mathrm{L}^2(\mathbb{R}))$.

(i)* Show that for two Schwartz functions $f, g \in \mathscr{S}(\mathbb{R})$ the convolution $f * g$ is also contained in $\mathscr{S}(\mathbb{R})$ and satisfies (2*)

$$\mathscr{F}(f * g) = \sqrt{2\pi} \cdot \mathscr{F} f \cdot \mathscr{F} g.$$

(ii)* Show that the function $\gamma \in \mathscr{S}(\mathbb{R})$ given by $\gamma(x) = e^{-\frac{x^2}{2}}$ for $x \in \mathbb{R}$ is a fixed point of $\mathscr{F}$, i.e., $\mathscr{F}\gamma = \gamma$. You may use that $\gamma$ is a Schwartz function and that (2*)

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \gamma(x) \, \mathrm{d}x = 1$$

holds. (Hint: Consider the linear ordinary differential equation $y'(x) + xy(x) = 0$ for $x \in \mathbb{R}$ and show that $\gamma$ and $\mathscr{F}\gamma$ both solve this equation with inital condition $y(0) = 1$. Uniqueness of the solution of this initial value problem then implies the claim.)

The *Gauss semigroup* $T$ on $\mathrm{L}^2(\mathbb{R})$ is defined by

$$T(t)f(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) \, \mathrm{d}y$$

for $x \in \mathbb{R}$, $f \in \mathrm{L}^2(\mathbb{R})$ and $t > 0$ as well as $T(0) := I$. Consequently, for each $t > 0$ we have $T(t)f = k_t * f$ for $f \in \mathrm{L}^2(\mathbb{R})$, if we set

$$k_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

for all $x \in \mathbb{R}$.

(iii) Show that $T(t) \in \mathscr{L}(\mathrm{L}^2(\mathbb{R}))$ for each $t > 0$. (1)

(iv) Show that the semigroup given by $S(t) := \mathscr{F} T(t) \mathscr{F}^{-1}$ for $t \geq 0$ is the diagonal semigroup on $\mathrm{L}^2(\mathbb{R})$ induced by the function (2)

$$q\colon \mathbb{R} \longrightarrow \mathbb{C}, \quad x \mapsto -x^2.$$

(Hint: Show this on the dense subspace $\mathscr{F}\mathscr{S}(\mathbb{R}) = \mathscr{S}(\mathbb{R})$ first (as in the lecture) using (i)* and (ii)*. Then use (iii) to show equality on all of $\mathrm{L}^2(\mathbb{R})$.)

(v) Conclude that $T$ is a $C_0$-semigroup with generator $A$ where (2)

$$D(A) = \mathrm{W}^{2,2}(\mathbb{R}), \quad Af = f'' \text{ for all } f \in D(A).$$