Universität Ulm

Submission: Friday, 19.05.2017

Prof. Dr. Wolfgang Arendt
Henrik Kreidler
Sommersemester 2017
Points total: 16 + 2*

Evolutionsgleichungen: Exercise Sheet 4

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. Let $q\colon \mathbb{R} \longrightarrow \mathbb{C}$ be continuous and M_q the associated *multiplication operator* on $\mathrm{C}_0(\mathbb{R})$, i.e.,

$$D(M_q) := \{f \in C_0(\mathbb{R})\colon qf \in \mathrm{C}_0(\mathbb{R})\},$$
$$M_q f := qf \text{ for all} f \in D(M_q).$$

(i) Show that $\sigma(M_q) = \overline{q(\mathbb{R})}$. (2*)
(Hint: Since $\lambda - M_q = M_{\lambda - q}$ for $\lambda \in \mathbb{C}$ it suffices to show that $0 \notin \overline{q(M)}$ if and only if M_q is invertible with bounded inverse. Show – by looking at suitable functions – that the existence of a bounded inverse of M_q implies that $|q|$ has a lower bound.)

(ii) Apply the Hille-Yosida theorem to show that $(M_q, D(M_q))$ generates a contractive C_0-semigroup if and only if $\operatorname{Re} q(x) \leq 0$ for all $x \in \mathbb{R}$. (4)

2. Consider the operator $(A, D(A))$ on $\mathrm{C}_0((0,1))$, where (4)

$$\mathrm{C}_0((0,1)) := \{f \in \mathrm{C}((0,1))\colon \forall \varepsilon > 0\, \exists \delta \in (0,1) \text{ with } |f(x)| \leq \varepsilon \text{ for } x \notin [\delta, 1-\delta]\}$$

as well as $D(A) := \{f \in \mathrm{C}^2([0,1])\colon f, f'' \in \mathrm{C}_0((0,1))\}$ and $Af := f''$ for $f \in D(A)$. Use the Lumer-Phillips theorem to show that $(A, D(A))$ generates a contractive C_0-semigroup on $\mathrm{C}_0((0,1))$. (Hint: To see that the operator is dissipative, choose $x \in (0,1)$ with $|f(x)| = \|f\|$. Then find a suitable $\alpha \in \mathbb{C}$ with $\mu := \alpha \cdot \delta_x \in J(f)$ and $\operatorname{Re}\langle Af, \mu\rangle \leq 0$; here δ_x denotes the evaluation in x.)

3. Let $\Omega \subseteq \mathbb{R}^d$ be open and (4)

$$\mathrm{H}_0^1(\Omega, \mathbb{R}) := \overline{\mathrm{C}_c^\infty(\Omega, \mathbb{R})}^{\mathrm{H}^1(\Omega,\mathbb{R})} \subseteq \mathrm{H}^1(\Omega, \mathbb{R}).$$

For $i, j \in \{1, ..., d\}$ let $a_{ij} \in \mathrm{L}^\infty(\Omega, \mathbb{R})$ with $a_{ij} = a_{ji}$ for all $i, j \in \{1, ..., d\}$ and assume that there is $\alpha > 0$ with

$$\sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq \alpha \|\xi\|_2^2$$

for all $\xi \in \mathbb{R}^d$ and almost every $x \in \Omega$. We consider the *elliptic operator* $(A, D(A))$ given by

$$D(A) := \left\{ u \in \mathrm{H}_0^1(\Omega)\colon \sum_{i,j} D_i(a_{i,j} D_j u) \in \mathrm{L}^2(\Omega, \mathbb{R}) \right\},$$
$$Au := \sum_{i,j} D_i(a_{i,j} D_j u) \text{ for } u \in D(A).$$

A reminder: Recall that by the lecture

$$\sum_{i,j} D_i(a_{i,j} D_j u) \in \mathrm{L}^2(\Omega, \mathbb{R})$$

means that there is $f \in \mathrm{L}^2(\Omega, \mathbb{R})$ with

$$-\int_\Omega \sum_{i,j} (a_{i,j} D_j u) D_i \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x$$

for all $\varphi \in \mathrm{C}_c^\infty(\Omega, \mathbb{R})$ and in this case we set

$$\sum_{i,j} D_i(a_{i,j} D_j u) := f.$$

Use the Lumer-Phillips theorem to show that $(A, D(A))$ generates a contractive C_0-semigroup on $\mathrm{L}^2(\Omega, \mathbb{R})$.
(Hint: Show that

$$[u, v] := \int_\Omega uv \,\mathrm{d}x + \sum_{i,j} \int_\Omega a_{ij} D_i u D_j v \,\mathrm{d}x$$

for $u, v \in \mathrm{H}_0^1(\Omega, \mathbb{R})$ defines an inner product on $\mathrm{H}_0^1(\Omega, \mathbb{R})$, which is equivalent to the ordinary inner product $(\,\cdot\,, \,\cdot\,)$ on $\mathrm{H}_0^1(\Omega, \mathbb{R})$, i.e., there are $\alpha, \beta > 0$ with

$$\alpha(u, u) \leq [u, u] \leq \beta(u, u)$$

for all $u \in \mathrm{H}_0^1(\Omega, \mathbb{R})$. Then use (as in the lecture) the Riesz-Fréchet theorem for the Hilbert space $(\mathrm{H}_0^1(\Omega, \mathbb{R}), [\,\cdot\,, \,\cdot\,])$ in order to show that $I - A$ is surjective.)

4. Let $(A, D(A))$ be a densely defined operator on a Banach space X and $(A', D(A'))$ its adjoint. Show that if $\lambda \in \varrho(A)$, then also $\lambda \in \varrho(A')$ and $R(\lambda, A)' = R(\lambda, A')$. (4)