Universität Ulm
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Evolutionsgleichungen: Exercise Sheet 5

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. Let $(A, D(A))$ be an operator on a Banach space X, $\lambda \in \varrho(A)$ and $D_0 \subseteq D(A)$ a subspace. Show that the following assertions are equivalent. (4)

(a) D_0 is dense in $D(A)$ with respect to the graph norm.
(b) $(\lambda - A)D_0$ is dense in X.
(c) $A = \overline{A|_{D_0}}$.

2. Let T be a C_0-semigroup with generator $(A, D(A))$ on a Banach space X and equip $D(A)$ with the graph norm. Show that the restriction (4)

$$T_1(t) := T(t)|_{D(A)} \text{ for } t \geq 0$$

defines a C_0-semigroup with generator $(A_1, D(A_1))$ where

$$D(A_1) = D(A^2),$$
$$A_1 x = Ax \text{ for all } x \in D(A_1).$$

3. For $1 \leq p \leq \infty$ we define

$$\mathrm{L}^p_\mathrm{c}(\mathbb{R}^d) := \{f \in \mathrm{L}^p(\mathbb{R}^d) \colon \exists K \subseteq \mathbb{R}^d \text{ compact with } f(x) = 0 \text{ for almost every } x \in \mathbb{R}^d \setminus K\}.$$

For conjugate indices $1 < p, q < \infty$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$) we define the *convolution*

(a) of $f \in \mathrm{L}^p(\mathbb{R}^d)$ and $g \in \mathrm{L}^q(\mathbb{R}^d)$
(b) of $f \in \mathrm{L}^p_\mathrm{loc}(\mathbb{R}^d)$ and $q \in \mathrm{L}^q_\mathrm{c}(\mathbb{R}^d)$

by

$$f * g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) \,\mathrm{d}y$$

for $x \in \mathbb{R}^d$.

(i) Show that for conjugate indices $1 < p, q < \infty$ one has (2*)

$$\mathrm{L}^p_\mathrm{loc}(\mathbb{R}^d) * \mathrm{L}^q_\mathrm{c}(\mathbb{R}^d) \subseteq \mathrm{C}(\mathbb{R}^d),$$
$$\mathrm{L}^p(\mathbb{R}^d) * \mathrm{L}^q(\mathbb{R}^d) \subseteq \mathrm{C}_0(\mathbb{R}^d).$$

(Hint: For the second inclusion use that the space $\mathrm{C}_c(\mathbb{R}^d)$ is dense in $\mathrm{L}^r(\mathbb{R}^d)$ for $1 \leq r < \infty$.)

(ii) Let $d \geq 3$ and let E be the *Newton potential*. Show that E is contained in $\mathrm{L}^q_\mathrm{loc}(\mathbb{R}^d)$ if and only if $q < \frac{d}{d-2}$. (2*)
To prove this, use the following result.
Let $0 \leq R_1 < R_2$ and consider

$$\Omega := \{x \in \mathbb{R}^d \colon R_1 < \|x\| < R_2\}.$$

For each measurable function $g \colon (R_1, R_2) \longrightarrow \mathbb{R}$ and the corresponding *radial function*

$$f \colon \Omega \longrightarrow \mathbb{R}, \quad x \mapsto g(\|x\|)$$

we obtain

$$\int_\Omega f(x)\,\mathrm{d}x = \sigma_d \int_{R_1}^{R_2} g(r) r^{d-1}\,\mathrm{d}r,$$

where σ_d is the surface volume of $\mathrm{S}^{d-1} = \{x \in \mathbb{R}^d \colon \|x\| = 1\}$.

(iii) Now let $d \geq 3$, $p > \frac{d}{2}$ and $f \in \mathrm{L}^p_\mathrm{c}(\mathbb{R}^d)$. Show that $E * f \in \mathrm{C}(\mathbb{R}^d)$. (1*)

Let $\mathbb{K} = \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^d$ be open, bounded and Dirichtlet regular with $d \geq 3$. Let further $p > \frac{d}{2}$ and let $m \colon \Omega \longrightarrow (0, \infty)$ be measurable with $\frac{1}{m} \in \mathrm{L}^p(\Omega)$. We consider $(A, D(A))$ on $\mathrm{C}_0(\Omega)$ with

$$D(A) := \left\{ u \in \mathrm{C}_0(\Omega) \colon \exists f \in \mathrm{C}_0(\Omega) \text{ with } \Delta u = \frac{f}{m} \right\},$$
$$Au := f \text{ für } u \in D(A) \text{ with } \Delta u = \frac{f}{m}.$$

(iv) Show that $(A, D(A))$ is closed. (2)

Now assume that m is continuous.

(v) Show that $(A, D(A))$ is dissipative. You may use that Lemma (11.4) holds for $u \in D(A)$, i.e., if $C := \max_{x \in \Omega} u(x) > 0$, then there is $x_0 \in \Omega$ with $u(x_0) = C$ and $\Delta u(x_0) \leq 0$. (We will discuss in the exercise group why this is correct.) (2*)

(vi) Show that $(A, D(A))$ is m-dissipative. (4)
(Hint: Show first that $(A, D(A))$ is surjective. Then prove that each dissipative, surjective and closed operator is already m-dissipative (cf. the proof of the *surjective Lumer-Phillips theorem* (10.3)).

(vii) Show that $(A, D(A))$ generates a contractive C_0-semigroup. (2)