Universität Ulm

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Points total: 16 +2*

Evolutionsgleichungen: Exercise Sheet 6

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. Let $(A, D(A))$ be an operator on a Banach space X with $\varrho(A) \neq \emptyset$. We equip $D(A)$ with the graph norm and consider the operator $(A_1, D(A_1))$ on $D(A)$, where (2*)

$$D(A_1) \coloneqq D(A^2),$$
$$A_1 x \coloneqq Ax \text{ for } x \in D(A_1).$$

Show that if $(A_1, D(A_1))$ generates a C_0-semigroup on $D(A)$, then $(A, D(A))$ generates a C_0-semigroup on X.

2. (i) Show with perturbation theory that $(A, D(A))$ with (2)

$$D(A) \coloneqq \{f \in \mathrm{C}^1(\mathbb{R}) \colon f, f' \in \mathrm{C}_0(\mathbb{R})\},$$
$$Af(x) \coloneqq f'(x) + f'(0)e^{-\frac{x^2}{2}} \text{ for } f \in D(A) \text{ and } x \in \mathbb{R},$$

generates a C_0-semigroup on $\mathrm{C}_0(\mathbb{R})$. You may use that $(B, D(B))$ with $D(B) \coloneqq D(A)$ and $Bf \coloneqq f'$ for $f \in D(B)$ is the generator of a C_0-semigroup (namely the shift semigroup).

(ii) For operators $(A, D(A))$ and $(B, D(B))$ on a Banach space X we define their *sum* $(A+B, D(A+B))$ by (2)

$$D(A+B) \coloneqq D(A) \cap D(B),$$
$$(A+B)x \coloneqq Ax + Bx \text{ for } x \in D(A+B).$$

Show with an example that the sum of two generators does not need to be a generator.

3. Let $(A, D(A))$ be the generator of a C_0-semigroup T on a Banach space X. Show that (4)

$$D(A^\infty) \coloneqq \{x \in X \colon x \in D(A^n) \text{ for all } n \in \mathbb{N}\}$$

is a core for A (i.e., $D(A^\infty)$ is dense in $D(A)$ with respect to the graph norm).
(Hint: Show first that $D(A^\infty)$ is dense in X and then apply a result of the lecture. For the density in X choose $\varphi \in \mathrm{C}_\mathrm{c}^\infty((0,\infty))$ with $\|\varphi\|_{\mathrm{L}^1((0,\infty))} = 1$ and for $x \in X$ look at the sequence $(x_n)_{n\in\mathbb{N}}$, where

$$x_n \coloneqq \int_0^\infty n\varphi(nt)T(t)x \,\mathrm{d}t$$

for $n \in \mathbb{N}$.)

4. Let $(A, D(A))$ be an operator on a complex Hilbert space H. Show that the following assertions are equivalent. (4)

(a) A is selfadjoint.

(b) A is densely defined, symmetric, closed and $\pm i - A^*$ is injective.

5. Let $(A, D(A))$ be a densely defined, symmetric operator on a complex Hilbert space H. Show that the following assertions are equivalent. (4)

(a) $\overline{A}$ is selfadjoint.

(b) $(\pm i - A)D(A)$ is dense in H.