Universität Ulm

Submission: Friday, 09.06.2017

Prof. Dr. Wolfgang Arendt
Henrik Kreidler
Sommersemester 2017
Points total: 14 +6*

Evolutionsgleichungen: Exercise Sheet 7

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. (i) Show that the numerical range of (2)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in \mathscr{L}(\mathbb{C}^2)$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$ is the line segment between λ_1 and λ_2.
Remark: With the spectral theorem we thus know the numerical range of arbitrary normal 2×2 – matrices.

(ii) Show that the numerical range of (2)

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in \mathscr{L}(\mathbb{C}^2)$$

is the closed unit disc in $\mathbb{C}$.
(Hint: Each $z \in \mathbb{C}$ can be written as $re^{i\theta}$ with $r \geq 0$ and $\theta \in \mathbb{R}$. Apply this to the coordinates of $v \in \mathbb{C}^2$ with $\|v\| = 1$.)

(iii) Consider the complex Hilbert space $H = \ell^2$ and $A(x_n)_{n\in\mathbb{N}} \coloneqq (x_{n+1})_{n\in\mathbb{N}}$ for $(x_n)_{n\in\mathbb{N}} \in \ell^2$. Show that $W(A)$ is the open unit disc in $\mathbb{C}$. (2)

2. Let H be a complex Hilbert space and $A \in \mathscr{L}(H)$ a bounded operator. Show that $\sigma(A) \subseteq \overline{\mathrm{W}(A)}$. (4)

3. Consider the complex Hilbert space $H = \mathrm{L}^2((0,\infty))$ and define operators $(A, D(A))$ and $(B, D(B))$ by

$$D(A) \coloneqq \mathrm{H}^1_0((0,\infty)), \quad Af \coloneqq -f' \text{ for all } f \in D(A),$$
$$D(B) \coloneqq \mathrm{H}^1((0,\infty)), \quad Bf \coloneqq f' \text{ for all } f \in D(B).$$

Then $(D, D(B))$ generates a contractive C_0-semigroup on H (namely the left shift; this is not to be shown).

(i) Show that $B = A^*$. In particular iA is symmetric. (2)

(ii) Show that $(A, D(A))$ generates a contractive C_0-semigroup. In particular we have (4*)

$$\sigma(A) \subseteq \mathbb{C}_- \coloneqq \{z \in \mathbb{C} \colon \operatorname{Re} z \leq 0\}.$$

(iii) Show that $\sigma(A) = \mathbb{C}_-$. (2*)
(Hint: It suffices to show that for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ the operator $\lambda - A^*$ is not injective, since this implies that $\lambda - A$ is not surjective.)

(iv) Conclude that $\overline{W(iA)} = \mathbb{R}$. In particular we obtain (2)

$$\sigma(iA) = i\mathbb{C}_- \not\subseteq \mathbb{R} = \overline{W(iA)}.$$