Universität Ulm

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Points total: 16 +4*

Evolutionsgleichungen: Exercise Sheet 8

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. Let $\theta \in (0, \frac{\pi}{2})$ and $T\colon \Sigma_\theta \longrightarrow \mathscr{L}(X)$ a holomorphic C_0-semigroup with generator A on a Banach space X. Show that T has a unique strongly continuous extension $\tilde{T}$ to $\overline{\Sigma_\theta}$. Show also that $S_\pm(t) \coloneqq \tilde{T}(e^{\pm i\theta}t)$ for $t \geq 0$ defines C_0-semigroups with generators $A_\pm$ where $A_\pm = e^{\pm i\theta}A$. (4*)

2. (i) We consider the Banach space $X = \mathrm{C}_0(\mathbb{R})$ and define the operator $(A, D(A))$ on X by (4)

$$D(A) \coloneqq \{f \in \mathrm{C}^1(\mathbb{R})\colon f, f' \in \mathrm{C}_0(\mathbb{R})\}, \quad Af \coloneqq f' \text{ for each } f \in D(A).$$

Show that $(A, D(A))$ does **not** generate a holomorphic C_0-semigroup on X.
(Hint: Show that $i\lambda \in \sigma(A)$ for all $\lambda \in \mathbb{R}$. To do this, find $f_n \in D(A)$ with $\|f_n\| = 1$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty}(i\lambda - A)f_n = 0$.)
Remark: The shift semigroup on $\mathrm{C}_0(\mathbb{R})$ therefore not a holomorphic contraction semigroup.

(ii) We consider the complex Hilbert space $H = \mathrm{L}^2(\mathbb{R})$ and the operator $(A, D(A))$ on H with (4)

$$D(A) = \mathrm{H}^2(\mathbb{R}) = \mathrm{W}^{2,2}(\mathbb{R}), \quad Af \coloneqq f'' \text{ für alle } f \in D(A).$$

Show that $(A, D(A))$ generates a holomorphic contraction semigroup on H.
Remark: Consequently, the Gauß semigroup on $\mathrm{L}^2(\mathbb{R})$ is a holomorphic contraction semigroup.

3. Let V be a complex Hilbert space. A sesquilinear form $a\colon V \times V \longrightarrow \mathbb{C}$ is called *continuous* if there is $M \geq 0$ with

$$|a(x,y)| \leq M \cdot \|x\| \cdot \|y\|$$

for all $x, y \in V$.
For a continuous sesquilinear form $a\colon V \times V \longrightarrow \mathbb{C}$ we consider

$$A_a\colon V \longrightarrow V, \quad x \mapsto A_a x$$

where $A_a x \in V$ is the (unique) element in V for $x \in V$ which satisfies $(A_a x|z) = a(x,z)$ for all $z \in V$.
For a bounded operator $A \in \mathscr{L}(V)$ we define $a_A(x,y) \coloneqq (Ax|y)$ for all $x, y \in V$.

(i) Show that we always have $A_a \in \mathscr{L}(V)$ and that a_A is always sesquilinear and continuous. Show also that $A_{a_A} = A$ and $a_{A_a} = a$ and that a is is sectorial if and only if A_a is sectorial. (Recall that a is called *sectorial* if there is $\theta \in [0, \frac{\pi}{2})$ such that $a(x) \coloneqq a(x,x) \in \overline{\Sigma_\theta}$ for all $x \in V$.) (2)

(ii) Show that there exists $\omega \in \mathbb{R}$ such that $A_a - \omega$ is sectorial. (2)

4. Let again V be a complex Hilbert space. A sesquilinear form $a\colon V \times V \longrightarrow \mathbb{C}$ is *coercive* if there is $\alpha > 0$ with $\operatorname{Re} a(x) \geq \alpha\|x\|^2$ for all $x \in V$. Now let a be a coercive and continuous sesquilinear form on V.

(i) Show that a is sectorial. (2)

(ii) Define $\|x\|_1 \coloneqq \sqrt{\operatorname{Re} a(x)}$ for $x \in V$. Show that $\|\cdot\|_1$ is a norm on V and that it is equivalent to the given norm $\|\cdot\|_V$ on V. (1)

(iii) We now also assume that V is a subspace of a Hilbert space H (with a different norm) and that there is $M \geq 0$ with $\|x\|_H \leq M \cdot \|x\|_V$ for all $x \in V$. Show that the norm $\|\cdot\|_a$ induced by a on V defined by (1)

$$\|x\|_a := (\operatorname{Re} a(x) + \|x\|_H^2)^{\frac{1}{2}}$$

for $x \in V$ is equivalent to $\|\cdot\|_1$.
Remark: This shows in particular that the form $a$is closed, i.e., V is complete with respect to $\|\cdot\|_a$.