Universität Ulm

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Prof. Dr. Wolfgang Arendt
Henrik Kreidler
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Points total: 16 +8*

Evolutionsgleichungen: Exercise Sheet 9

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. For this exercise we use the fact that each $f \in \mathrm{H}^1((a,b))$ can be identified with a function in $\mathrm{C}([a,b])$ and that

$$f(x) = f(a) + \int_a^x f'(y)\,\mathrm{d}y$$

for all $x \in [a,b]$. (Each $f \in \mathrm{H}^1((a,b))$ is an equivalence class of functions. One can show that this equivalence class contains exactly one function in $\mathrm{C}([a,b])$ and then identify f with it.)
Now let $b > 0$ and $V \coloneqq \{u \in \mathrm{H}^1((0,b)) \colon u(0) = 0\}$.

(i) Show the *Poincaré inequality* (2)

$$\int_0^b |u(x)|^2\,\mathrm{d}x \leq \frac{b^2}{2}\int_0^b |u'(x)|^2\,\mathrm{d}x \text{ for all } u \in V.$$

(ii) Show that the sesquilinear form (2)

$$a\colon V \times V \longrightarrow \mathbb{C}, \quad (u,v) \mapsto \int_0^b u'(x)\overline{v'(x)}\,\mathrm{d}x$$

is continuous and coercive.

(iii) We now conisder V as a subspace of $H \coloneqq \mathrm{L}^2((0,b))$. Show that the for a on H is sectorial, closed and densely defined. (2)
(Hint: Use exercise 4 of exercise sheet 8.)

(iv) Show that for $f, g \in \mathrm{H}^1((0,b))$ we have (2)

$$\int_0^b f'(x)g(x) + f(x)g'(x)\,\mathrm{d}x = f(b)g(b) - f(0)g(0).$$

(v) Show that the associtaed operator A with a is given by (2)

$$D(A) = \{u \in \mathrm{H}^2((0,b)) \colon u(0) = 0, u'(b) = 0\},$$
$$Au = -u'' \text{ for all } u \in D(A).$$

Here

$$\mathrm{H}^2((0,b)) = \{f \in \mathrm{H}^1((0,b)) \colon f' \in \mathrm{H}^1((0,b))\}$$

and $f'' \coloneqq (f')'$ for $f \in \mathrm{H}^2((0,b))$.
Anmerkung: By the results of the lecture, $-A$ generates a holomorphic contraction semigroup on $\mathrm{L}^2((0,b))$.

2. (i) Let A generate a strongly continuous **group** on a Banach space X. Show that A^2 generates a bounded holomorphic C_0-semigroup on X. (4)
(Hint: Let $\theta \in (0, \frac{\pi}{2})$ be arbitrary and $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$. We find a root $\mu = re^{i\alpha}$ of λ with $|\alpha| < \frac{1}{2}(\frac{\pi}{2} + \theta)$. Write the operator $(\lambda - A^2)$ as a product and show that $\lambda \in \varrho(A^2)$ and

$$R(\lambda, A^2) = R(\mu, A)R(\mu, -A).$$

Then use this identity as well as estimates for the resolvents of A and $-A$ to verify condition (i) in theorem (20.3).)
Remark: The result proved here extends to the unbounded case: If A is the generator of any C_0-group, then A^2 generates a holomorphic C_0-semigroup.

(ii) We consider the operator A om $\mathrm{C}_0(\mathbb{R})$ given by (2)

$$\begin{aligned} D(A) &:= \{f \in \mathrm{C}^2(\mathbb{R}) \colon f, f', f'' \in \mathrm{C}_0(\mathbb{R})\}, \\ Af &:= f'' \text{ for all } f \in D(A). \end{aligned}$$

Show that A generates a holomorphic C_0-semigroup.

3. We consider the following version of the so called *Black-Scholes equation* which describes the value of call-options at the European stock market.

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = -\frac{\sigma^2}{2} x^2 \frac{\partial^2 u}{\partial x^2}(x,t) - rx\frac{\partial u}{\partial x}(x,t) + ru(x,t) & \text{for all } x \in (0,\infty),\ t \in [0,T], \\ u(x,T) = h(x) & \text{for all } x \in (0,\infty), \end{cases}$$

where $\sigma > 0$ (volatility), $r > 0$ (interest rate) and $T > 0$ (tenor) are fixed parameters and $h \in \mathrm{C}_0((0,\infty))$ a fixed function. By substitution and rescaling this problem can be described by the abstract Cauchy problem

$$\begin{aligned} f'(t) &= Af(t) \text{ for all } t \geq 0, \\ f(0) &= h \end{aligned}$$

on the space $\mathrm{C}_0((0,\infty))$ where

$$\begin{aligned} D(A) &:= \{f \in \mathrm{C}^2((0,\infty)) \colon q^2 \cdot f'',\ q \cdot f',\ f \in \mathrm{C}_0((0,\infty))\}, \\ Af &:= q^2 \cdot f'' + c \cdot q \cdot f' - cf \text{ for } f \in D(A) \end{aligned}$$

and $q(x) := x$ for all $x \in (0,\infty)$ as well as $c := \frac{2r}{\sigma^2} > 0$. We want to show that $(A, D(A))$ generates a holomorphic C_0-semigroup.

(i) Let $\eta := \frac{1}{2}(c-1)$ and define $(B, D(B))$ by (2*)

$$\begin{aligned} D(B) &:= \{f \in \mathrm{C}^1((0,\infty)) \colon q \cdot f', f \in \mathrm{C}_0((0,\infty))\}, \\ Bf &:= q \cdot f' \text{ for all } f \in D(B). \end{aligned}$$

Show that $D(A) = D(B^2)$ and

$$Af = (B+\eta)^2 f - (1+\eta)^2 f$$

for all $f \in D(A)$.

(ii) Show that $Vf(x) := f(e^x)$ for $f \in \mathrm{C}_0((0,\infty))$ and $x \in \mathbb{R}$ defines an isometric isomorphism (2*)

$$V \colon \mathrm{C}_0((0,\infty)) \longrightarrow \mathrm{C}_0(\mathbb{R}).$$

(iii) Show that the operator VBV^{-1} (with domain $\{f \in \mathrm{C}_0(\mathbb{R}) \colon V^{-1}f \in D(B)\}$) is the first derivative on $\mathrm{C}_0(\mathbb{R})$ with domain (2*)

$$\{f \in \mathrm{C}^1(\mathbb{R}) \colon f, f' \in \mathrm{C}_0(\mathbb{R})\}.$$

(iv) Show that $(A, D(A))$ generates a holomorphic C_0-semigroup. You may use the generalized result of exercise 2(i). (2*)