Universität Ulm

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Evolutionsgleichungen: Exercise Sheet 10

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de.
Exercises marked with * are bonus exercises.

1. For $A \in \mathscr{L}(\mathbb{C}^2)$ let T be the associated semigroup, i.e., $T(t) := e^{tA}$ for all $t \geq 0$. Write down examples for matrices $A \neq I$ with the following properties. (5)

(i) $\lim_{t\to\infty} T(t)x = 0$ for each $x \in \mathbb{C}^2$.

(ii) T is *periodic*, i.e., there is $t > 0$ with $T(t) = I$.

(iii) For each $x \in \mathbb{C}^2 \setminus \{0\}$ the orbit $\{T(t)x \colon t \geq 0\}$ is unbounded.

(iv) There are $x, y \in \mathbb{C}^2 \setminus \{0\}$ such that $\{T(t)x \colon t \geq 0\}$ is bounded and $\{T(t)y \colon t \geq 0\}$ is unbounded.

(v) $\omega(A) = 0$, but T is unbounded.

2. A C_0-semigroup T is *uniformly stable* if $\lim_{t\to\infty} \|T(t)\| = 0$. Now let T be a C_0-semigroup with generator A on a Banach space X. Show that the following assertions are equivalent. (3)

(a) $\omega(A) < 0$.

(b) T is uniformly stable.

(c) There is $t > 0$ with $\|T(t)\| < 1$.

(d) There is $t > 0$ with $r(T(t)) < 1$.

3. A C_0-semigroup T with generator A satisfies the *weak spectral mapping theorem* if

$$\sigma(T(t)) = \overline{e^{t\sigma(A)}} \text{ for all } t \geq 0.$$

(i) Now let $q \colon \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with bounded real part T the associated multiplication semigroup on $\mathrm{C}_0(\mathbb{R})$, i.e., $T(t)f(x) := e^{tq(x)}f(x)$ for all $f \in \mathrm{C}_0(\mathbb{R})$, $x \in \mathbb{R}$ and $t \geq 0$ (see exercise 4 of exercise sheet 1). Show that T satisfies the weak spectral mapping theorem. (Hint: Use exercise 1 of exercise sheet 4.) (4)

(ii) Now let T be a C_0-semigroup with generator A which satisfies the spectral mapping theorem. Show that $s(A) = \omega(A)$. (4)

4. Let X and Y be Banach spaces and assume that Y is a subspace of X (with a possibly different norm). The inclusion $X \subseteq Y$ is called *continuous* if there is $M \geq 0$ with $\|y\|_X \leq M\|y\|_Y$ for every $y \in Y$.
Now let A be an operator on a Banach space X. Let Y be another Banach space with continuous inclusion $Y \subseteq X$ and let $A|_Y$ be the *part of A in Y*, i.e.,

$$D(A|_Y) := \{y \in D(A) \cap Y \colon Ay \in Y\},$$
$$A|_Y y := Ay \text{ for all } y \in D(A|_Y).$$

(i) Let $\lambda \in \varrho(A)$ with $R(\lambda, A)Y \subseteq Y$. Show that $\lambda \in \sigma(A|_Y)$ and $R(\lambda, A|_Y) = R(\lambda, A)|_Y$. (2*)

We now also assume that $\varrho(A) \neq \emptyset$, that $D(A)$ carries the graph norm and that $D(A) \subseteq Y$ continuously.

(ii) Show that A_1 (see exercise 1 of exercise sheet 6) is the part of $A|_Y$ in $D(A)$. (2*)

(iii) Show that $\sigma(A|_Y) = \sigma(A)$. (1*)
(Hint: Use (i) and (ii) in order to show $\sigma(A_1) \subseteq \sigma(A|_Y)$. We know from the proof of exercise 1 of exercise sheet 6 that A_1 and A are *similar*, i.e., there is an isomorphism $V \in \mathscr{L}(X, D(A))$ (namely the resolvent) with $A_1 = V^{-1}AV$. Since similar operators have the same spectrum (this is not to be shown here), we obtain $\sigma(A_1) = \sigma(A)$.)

5. Let $1 < p < \infty$. For each $q \in [p, \infty)$ we consider the space $X_q := \mathrm{L}^p((1,\infty)) \cap \mathrm{L}^q((1,\infty))$. By setting

$$\|f\| := \max(\|f\|_p, \|f\|_q)$$

for $f \in X_q$ we define a norm on X_q in relation to which X_q is complete. We now define $T_q(t)f(x) := f(x \cdot e^t)$ for $f \in X_q$, $x \in (1,\infty)$ and $t \geq 0$. Then T_q is a C_0-semigroup and let A_q be its generator.

(i) Show that $\omega(A_q) = -\frac{1}{q}$. (2*)
(Hint: Show first that $\|T_q(t)f\| \leq e^{-\frac{t}{q}}\|f\|$ for all $f \in X_q$ and $t \geq 0$. Now fix $t \geq 0$ and consider $f \in X_q$ given by

$$f(x) := \begin{cases} 1 & e^t \leq x \leq e^t + 1, \\ 0 & \text{else.} \end{cases}$$

and conclude that $\|T_q(t)\| = e^{-\frac{t}{q}}$ for all $t \geq 0$.)

(ii) Show that $s(A_q) \geq -\frac{1}{p}$. (2*)
(Hint: For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < -\frac{1}{p}$ consider the function f_λ defined by $f_\lambda(x) := x^\lambda$ for $x \in (1,\infty)$. Then show $f_\lambda \in X_q$ and $T(t)f_\lambda = e^{\lambda t} f_\lambda$ for all $t \geq 0$.)

(iii) Show that $s(A_q) = -\frac{1}{p}$. You may use that A_q is the part of A_p in X_q[1]. (7*)
(Hint: In case $p = q$ we now know by (i) and (ii) that $s(A_p) = \omega(A_p) = -\frac{1}{p}$. In particular, we obtain – since $\omega < 0$ – the representation of the resolvent

$$(R(0, A_p)f)(x) = \int_0^\infty (T_p(s)f)(x)\,\mathrm{d}s.$$

for almost all $x \in (1,\infty)$ for every $f \in X_p = \mathrm{L}^p((1,\infty))$. Now show with suitable estimates that

$$D(A_p) = \mathrm{Bild}(R(0, A_p)) \subseteq X_q \subseteq X_p = \mathrm{L}^p((1,\infty))$$

and that these inclusions are continuous. Then apply exercise 4.)

Remark: For $p < q$ we therefore obtain $\omega(A_q) < s(A_q)$.

[1] This is a consequence of the fact that T_q is the restriction of T_p to X_q, see for example section II.2.3 in Engel, Nagel: One-Parameter Semigroups for Linear Evolution Equations.