§20 The generation theorem for general holomorphic semigroups

(20.1) Theorem

Let A be an operator on a Banach space X with

(a) $\exists\, \theta \in (0, \frac{\pi}{2}]$: $\Sigma_{\theta+\frac{\pi}{2}} \subseteq \rho(A)$

(b) $\exists\, M \geq 1$: $\|\lambda R(\lambda, A)\| \leq M$. $\forall \lambda \in \Sigma_{\theta+\frac{\pi}{2}}$

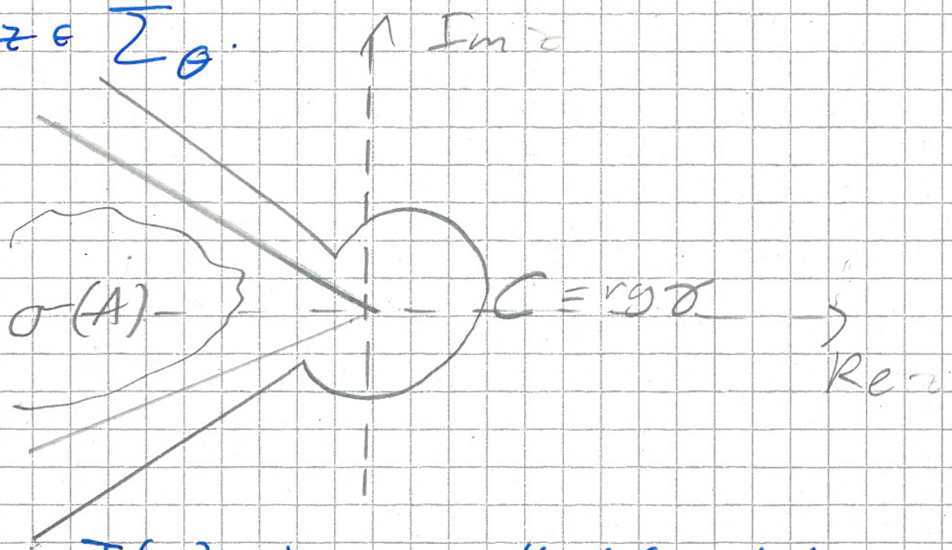
Define $T(0) := I$ and

$$T(z) := \frac{1}{2\pi i}\int_C e^{\mu z} R(\mu, A)\, d\mu$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{\gamma(t) z} R(\gamma(t), A)\, \gamma'(t)\, dt$$

for a piecewise smooth curve C with parametrization $\gamma : \mathbb{R} \to C \subseteq \Sigma_{\theta+\frac{\pi}{2}}$ satisfying

* $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$ and
* $\lim_{t\to\pm\infty} \frac{\gamma(t)}{|\gamma(t)|} = e^{\pm i(\frac{\pi}{2}+\delta)}$ for some $\delta \in (|\arg z|, \theta)$

and $z \in \Sigma_\theta$.



Then: $T(z)$ is a well-defined bounded operator $\forall z \in \Sigma_\theta$ and the definition does not depend ~~from~~ on C

Moreover:

(i) $\forall \tilde\theta \in (0,\theta)$: $\|T(z)\|$ is uniformly bounded on $\Sigma_{\tilde\theta}$

(ii) $z \longmapsto T(z)$ is ~~analytic~~ holomorphic on Σ_θ

(iii) $T(z_1+z_2) = T(z_1)\,T(z_2) \quad \forall z_1, z_2 \in \Sigma_\theta$.

Proof (partial):

Let $\tilde\theta \in (0,\theta)$ and set $\varepsilon := \frac{\theta - \tilde\theta}{2}$.

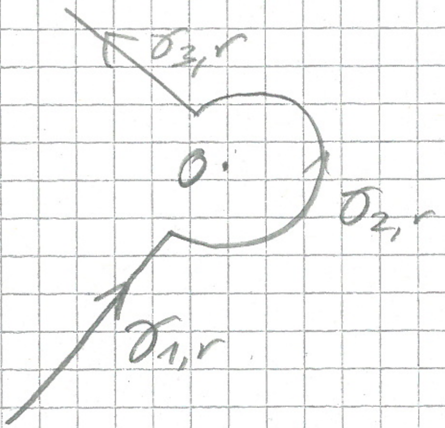
For $r>0$ we define

$$\gamma_{1,r}(t) := -t\, e^{-i(\frac{\pi}{2}+\tilde\theta+\varepsilon)} \quad \forall t \in (-\infty,-r),$$

$$\gamma_{2,r}(t) := r\, e^{it} \quad \forall t \in [-(\tfrac{\pi}{2}+\tilde\theta+\varepsilon), \tfrac{\pi}{2}+\tilde\theta+\varepsilon),$$

$$\gamma_{3,r}(t) := t\, e^{i(\frac{\pi}{2}+\tilde\theta+\varepsilon)} \quad \forall t \in (r,\infty)$$

Concatenation of the associated curves $C_{i,r}$ yields a curve C_r:



Now let $z \in \Sigma_{\tilde\theta}$. Then C_r is a curve as in the theorem if we set $r := \frac{1}{|z|}$.

In order to see that

$$\frac{1}{2\pi i}\int_{C_r} e^{\mu z} R(\mu, A)\, d\mu \qquad (*)$$

exists, we look at the integrals

$$\frac{1}{2\pi i}\int_{C_{i,r}} \| e^{\mu z} R(\mu, A)\|\, d\mu.$$

Take $\mu \in C_{3,r} = \mathrm{rg}\,\gamma_{3,r}$. $\tilde\theta \le \frac{\pi}{2}$

$$\Rightarrow \frac{\pi}{2}+\varepsilon \le \underbrace{\frac{\pi}{2}+\tilde\theta+\varepsilon}_{=\arg\mu} + \underbrace{\arg z}_{\in(-\tilde\theta,\tilde\theta)} \le \frac{3\pi}{2}-\varepsilon$$

$$\Rightarrow \frac{1}{|\mu z|}\operatorname{Re}(\mu z) = \operatorname{Re}\left(e^{i(\arg\mu+\arg z)}\right)$$

$$= \cos(\underbrace{\arg\mu+\arg z}_{\in(\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2}-\varepsilon)}) \le \cos\left(\tfrac{\pi}{2}+\varepsilon\right) = -\sin(\varepsilon)$$

$$\Rightarrow \int_{C_{3,r}} \| e^{\mu z} R(\mu,A)\|\,d\mu \overset{(b)}{\le} \int_{C_{3,r}} \frac{|e^{\mu z}|}{|\mu|}\cdot M\,d\mu$$

$$\le \int_{C_{3,r}} \frac{e^{-|\mu z|\sin(\varepsilon)}}{|\mu|}\cdot M\,d\mu = \int_r^\infty \frac{e^{-t|z|\sin(\varepsilon)}}{t}\cdot M\,dt$$

$s = t|z|$

$$\Rightarrow \int_{C_{3,r}} \| e^{\mu z} R(\mu,A)\|\,d\mu \le \int_1^\infty \frac{e^{-s\sin(\varepsilon)}}{s}\cdot M\,dt \qquad (**)$$

The same estimate holds for $C_{1,r}$.

Now take $\mu \in C_{2,r} = \mathrm{rg}\,\gamma_{2,r}$. $\Rightarrow \mu = re^{it}$ for some $t \in \mathbb{R}$.

$$\Rightarrow |e^{\mu z}| = e^{r\operatorname{Re} e^{it} z} \le e^{r\cdot|z|} = e.$$

$$\Rightarrow \int_{C_{2,r}} \| e^{\mu z} R(\mu,A)\|\,d\mu \le e\int_{-\frac{\pi}{2}-\tilde\theta-\varepsilon}^{\frac{\pi}{2}+\tilde\theta+\varepsilon} \frac{M}{r}\cdot r\,dt$$

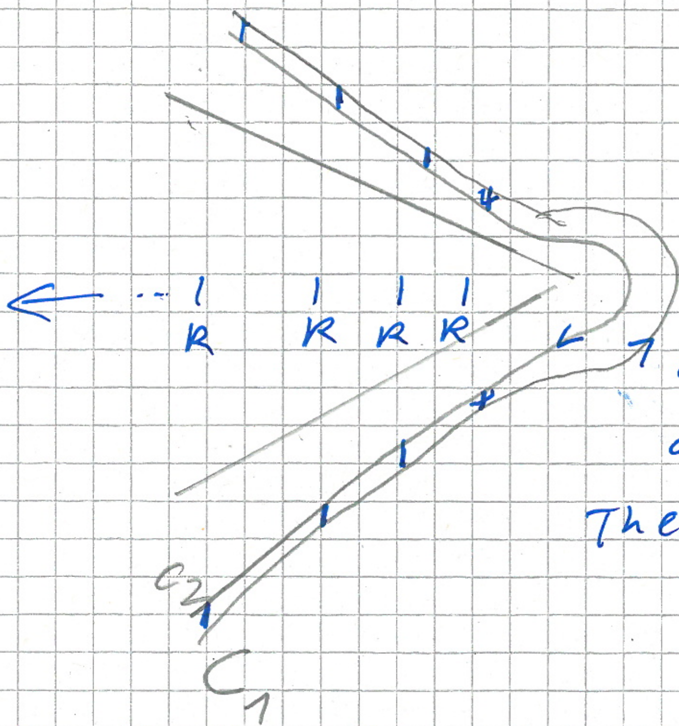
$$\Rightarrow \int_{C_{2,r}} \| e^{\mu z} R(\mu,A)\|\,d\mu \le 2\pi\cdot e\cdot M \qquad (***)$$

$(**) + (***)$ imply $T(z) \in \mathcal{L}(X)$ and uniform boundedness on $\Sigma_{\tilde\theta}$.

We omit the proof of (ii) + (iii) and

note that curve-independence follows from Cauchy's integral theorem (+ some thought). $\square$

curve independence (sketch)



By connecting C_1 and C_2 with vertical lines over $R \in (-\infty, 0)$ we obtain closed curves and can apply CIT. Then $R \to -\infty$.

(30.2) Proposition

Let A be an operator on a Banach space X with (a) + (b).
If A is densely defined, then the semigroup T of (30.1) is strongly continuous and A is its generator.

We omit the proof, see II.4.a of Engel-Nagel.

(30.3) Theorem

Let A be an operator on a Banach space X. The following are equivalent:

(i) $\exists\, \theta \in (0, \frac{\pi}{2}]$: $\Sigma_{\theta + \frac{\pi}{2}} \subseteq \rho(A)$,
$\exists\, M \geq 1$: $\| \lambda R(\lambda, A) \| \leq M \;\; \forall \lambda \in \Sigma_{\theta + \frac{\pi}{2}}$
and A is densely defined.

(ii) A generates a bounded holomorphic C_0-semigroup (i.e., a strongly cont. sgr.

with a bounded holomorphic extension to some sector Σ_{ϑ}).

(iii) A generates a bounded C_0-semigroup and $\exists C \geq 1$ with

$$\|s R(r+is, A)\| \leq C \quad \forall s \in \mathbb{R}, r > 0.$$

<u>Proof</u>: (20.1) + (20.2) $\Rightarrow$ „(i) $\Rightarrow$ (ii)".

„(ii) $\Rightarrow$ (iii)": For $\tilde{\theta} < \vartheta$ we obtain that $t \longmapsto T(e^{\pm i\tilde{\theta}} t)$ are bounded C_0-semigroups with generators $e^{\pm i\tilde{\theta}} A$. In particular there is $\tilde{C} \geq 1$ with

$$\|(\operatorname{Re}\lambda)\, R(\lambda, e^{\pm i\theta} A)\| \leq \tilde{C} \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

$$\Rightarrow \|R(r+is, A)\| = \|e^{-i\tilde{\theta}} R(e^{-i\tilde{\theta}}(r+is), e^{-i\tilde{\theta}} A)\| \leq \frac{\tilde{C}}{\operatorname{Re}(e^{-i\theta}(r+is))}$$

Writing $e^{-i\tilde{\theta}} = a + ib$, with $a, b > 0$ we obtain:

$$\operatorname{Re} e^{-i\theta}(r+is) = ar + sb \geq sb$$

and therefore

$$\|sR(r+is, A)\| \leq \frac{\tilde{C}}{b} \quad \forall s > 0.$$

The same estimate holds for $s < 0$.

„(iii) $\Rightarrow$ (i)":

Since A generates a bounded C_0-sgr. we have $\Sigma_{\frac{\pi}{2}} \subseteq \rho(A)$ and $\exists \tilde{M} \geq 1$:

$$(1) \qquad \|(\operatorname{Re}\lambda)\, R(\lambda, A)\| \leq \tilde{M} \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

By (3.5) we know

$$\|R(\lambda, A)\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \quad \forall \lambda \in \rho(A)$$

Combining this with (iii) we obtain

$$\operatorname{dist}(r+is, \sigma(A)) \geq \frac{|s|}{C} > 0 \quad \forall r>0, s \in \mathbb{R}\setminus\{0\}.$$

$$\Rightarrow \quad \operatorname{dist}(is, \sigma(A)) \geq \frac{|s|}{C} > 0 \quad \forall s \in \mathbb{R}\setminus\{0\}$$

$$\Rightarrow \quad i\mathbb{R}\setminus\{0\} \subseteq \varrho(A).$$

Since the resolvent map is continuous, (iii) implies

$$(2) \quad \|R(\mu, A)\| \leq \frac{C}{|\mu|} \quad \forall \mu \in i\mathbb{R}\setminus\{0\}.$$

Now let $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, $\operatorname{Re}\lambda \leq 0$ and $\left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| < \frac{1}{C}$. Then:

$$|\lambda - i\cdot\operatorname{Im}\lambda| \cdot \|R(i\cdot\operatorname{Im}\lambda, A)\| \underset{(2)}{\leq} |\operatorname{Re}\lambda| \cdot \frac{C}{|\operatorname{Im}\lambda|} < 1.$$

$$\overset{(3.3)}{\Rightarrow} \quad \lambda \in \varrho(A) \text{ and } R(\lambda, A) = \sum_{n=0}^{\infty} (-\operatorname{Re}\lambda)^n R(i\operatorname{Im}\lambda, A)^{n+1}$$

Setting $\theta := \arctan\frac{1}{C}$ we therefore obtain:

$$\Sigma_{\frac{\pi}{2}+\theta} = \Sigma_{\frac{\pi}{2}} \cup \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0, \operatorname{Im}\lambda \neq 0, \left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| < \frac{1}{C}\right\} \subseteq \varrho(A).$$

We now have to verify an estimate as in (b) of (20.1).

By (iii) we have

$$(3) \quad \|(\operatorname{Im}\lambda) R(\lambda, A)\| \leq C \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

Equations (1) + (3) imply

$$\|\lambda R(\lambda, A)\| \leq 2\cdot\max\{\hat{\mu}, C\} \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \leq 0$, $\mathrm{Im}\,\lambda \neq 0$
and $\left|\frac{\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda}\right| \leq \frac{q}{C}$, $q<1$, we obtain

$$\|R(\lambda, A)\| \leq \sum_{n=0}^{\infty} |\mathrm{Re}\,\lambda|^n \|R(i\,\mathrm{Im}\,\lambda, A)^{n+1}\|$$
$$\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{(\mathrm{Im}\,\lambda)^{n+1}} \cdot |\mathrm{Re}\,\lambda|^n$$
$$\leq \sum_{n=0}^{\infty} q^n \frac{C^n}{|\mathrm{Im}\,\lambda|} = \frac{1}{1-q} \frac{C}{|\mathrm{Im}\,\lambda|}$$

and since

$$C^2 \cdot \frac{|\lambda|^2}{|\mathrm{Im}\,\lambda|^2} = \frac{C^2 |\mathrm{Re}\,\lambda|^2 + C^2 |\mathrm{Im}\,\lambda|^2}{(\mathrm{Im}\,\lambda)^2}$$
$$< 1 + C^2$$

we have:

$$\|\lambda R(\lambda, A)\| \leq \frac{1}{1-q} \sqrt{1+C^2} \quad \forall \lambda \in \Sigma_{\tilde{\theta}}$$

with $\tilde{\theta} := \arctan\left(\frac{q}{C}\right) < \theta$. $\square$

By rescalling we obtain.

(20.4) <u>Corollary</u>

Let A be an operator on a Banach space X. The following are equivalent:

(i) A is d.d., $\exists\, \theta \in (0, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$, $M \geq 1$:
$\omega + \Sigma_{\theta + \frac{\pi}{2}} \subseteq \varrho(A)$ and

$$\|(\lambda-\omega) R(\lambda, A)\| \leq M \quad \forall \lambda \in \omega + \Sigma_{\theta+\frac{\pi}{2}}$$

(ii) A generates a holomorphic C_0-semigroup

(iii) A generates a C_0-sgr. T with $\|T(t)\| \leq M e^{\omega t}$ $\forall t \geq 0$ and
$\exists\, C \geq 1$: $\|s R(r+\omega+is, A)\| \leq C$ $\forall s \in \mathbb{R}$, $r > 0$.