Semigroups and Evolution Equations

SS 2018

References: Pazy

Engel Nagel

W. A. , Heat Kernels.

Chapter 1. Semigroups and their generators

§ 1 C_0-semigroups.

X Banach space.

$\mathcal{L}(X) := \{ S : X \to X : \text{linear, continuous} \}$

$\| S \| = \sup_{\|x\| \le 1} \| Sx \|$

$\mathcal{L}(X)$ is a Banach space

$S_n \to S$ in $\mathcal{L}(X)$ $:\Leftrightarrow$ $\| S_n - S \| \to 0$

convergence in operator norm

$S_n \longrightarrow S$ <u>strongly</u> $:\Longleftrightarrow$ $S_n x \longrightarrow Sx$ in X
$\forall x \in X$

(this is much weaker).

$\mathcal{L}(X)$ is a <u>Banach algebra</u>:

$S_1, S_2 \in \mathcal{L}(X) \Longrightarrow S_1 \circ S_2 \in \mathcal{L}(X)$ &
$\|S_1 S_2\| \leq \|S_1\| \, \|S_2\|$.

(1.1) <u>Definition.</u> A <u>C_0-semigroup</u> on X is a mapping $T : (0, \infty) \to \mathcal{L}(X)$ such that

(a) $T(t+s) = T(t) \, T(s)$

(b) $\lim\limits_{t \downarrow 0} T(t)x = x \quad \forall x \in X$

C_0-semigroup $=$ <u>strongly continuous semigroup</u>

Frequently $T(0) := I$ $\quad (T(t))_{t \geq 0} = T$.

(1.2) Properties: 1. ~~1.~~ Put $T(0) = I$

Then $T(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is

strongly continuous

2. $\exists\, M, \omega$ $\quad \| T(t) \| \le M e^{\omega t} \quad (t \ge 0)$

(at most exponential growth).

Recall: uniform boundedness principal.

Proof. a) $\exists\, \tau > 0$ $\quad M = \sup\limits_{0 < t \le \tau} \| T(t) \| < \infty$.

Otherwise, $\exists\, t_n \downarrow 0$ $\quad \| T(t_n) \| \to \infty$.

UBP $\Rightarrow$ $\exists\, x$ $\quad \| T(t_n) x \| \to \infty$ $\quad$ ↯.

b) Let $t > 0$. $\exists!\, n \in \mathbb{N}_0$ $\quad t = n\tau + s$

$0 \le s < \tau$. $\quad \| T(t) \| = \| T(n\tau)\, T(s) \|$

$= \| T(\tau)^n\, T(s) \| \le M \cdot M^n$.

$\omega = \log M$. $\quad M^n = e^{\omega n} \le e^{\omega t}$.

Thus $\| T(t) \| \le M e^{\omega t}$

c) Let $x \in X$, $t > 0$

1 st case $t_n \downarrow t$ $\Rightarrow$ $T(t_n)x - T(t)x =$

$T(t_n - t)\, T(t)x - T(t)x \longrightarrow 0$.

2 nd case $t_n \uparrow t$.

$T(t_n)x - T(t)x = T(t_n)(x - T(t - t_n)x)$

$\longrightarrow 0$. $\square$

<u>Lemma.</u> $S_n \to S$ strongly

$x_n \longrightarrow x \Rightarrow S_n x_n \longrightarrow Sx$.

UBP

Proof. $\sup \| S_n \| < \infty$

$S_n x_n - Sx = S_n(x_n - x) + S_n x - Sx$ $\square$

<u>Examples of semigroups.</u>

(1.3) <u>Example.</u> $A \in \mathcal{L}(X)$

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

$\| e^{tA} - I \| \longrightarrow 0$ $(t \downarrow 0)$

(1.4) <u>Equicontinuity Lemma.</u>

$S_n \in \mathcal{L}(X, Y)$, $\|S_n\| \le M$.

Equivalent.

(i) $\exists X_0$ dense in X such that

$\lim S_n x$ exists $\forall x \in X$

(ii) $\exists S \in \mathcal{L}(X, Y)$ $S_n x \to S x$

uniformly on compact subsets.

<u>Proof.</u> Assume (i)

a) $\lim_{n\to\infty} S_n x =: Sx$ exists $\forall x \in X$

Let $x \in X$, $\varepsilon > 0$. $\exists x_0 \in X_0$ $\|x - x_0\| \le \varepsilon$.

$\exists n_0$ $\|S_n x_0 - S_m x_0\| \le \varepsilon$ $\forall n, m \ge n_0$

$\Rightarrow \|S_n x - S_m x\| \le \|S_n(x - x_0)\| + \|S_n x_0 - S_m x_0\|$

$\cancel{+\|(S_n - S_m)x_0\|}$ $\le M\varepsilon + \varepsilon$ $n, m \ge n_0$

$\Rightarrow (S_n x)$ CS

$\Rightarrow Sx := \lim S_n x$ exists for all $x \in X$. Clearly $S \in \mathcal{L}(X, Y)$

b) uniformly on compact subsets.

Let $K \subset X$ be compact, $\varepsilon > 0$

$\exists\ y_1, \dots, y_m \qquad K \subset \bigcup_{j=1}^{m} B(y_j, \varepsilon)$

$\exists\ n_0 \quad \forall n \geq n_0 \qquad \| S_n y_j - S y_j \| \leq \varepsilon \quad \forall n \geq n_0$
$\forall j = 1 \dots m$

Let $x \in K$. $\exists j \qquad \|x - y_j\| \leq \varepsilon . \quad \Rightarrow$

$$\| S_n x - S x \| \leq \| (S_n - S)(x - y_j) \| + \| S_n y_j - S y_j \|$$
$$\leq 2M\varepsilon + \varepsilon \qquad \forall n \geq n_0 \quad \square .$$

(1.5) ~~Co~~ Proposition. Let $T : (0, \infty) \to \mathcal{L}(X)$ be a semigroup such that $\|T(t)\| \leq M$ $0 < t \leq 1$. If $\exists\ X_0 \subset X$ dense such that $T(t) x_0 \to x_0 \ (t \downarrow 0) \ \forall x_0 \in X_0$, then ~~T(t)~~ T is a C_0-semigroup.

Rk. $T : (0, \infty) \to \mathcal{L}(X)$ semigroup $:\Leftrightarrow$

$T(t+s) = T(t) T(s)$.

Proof. Let $t_n \downarrow t \quad \Rightarrow \quad T(t_n) x_0 \to x_0 = I x_0$

$\forall x_0 \in X_0$ Equicontinuity lemma $\Rightarrow T(t_n) x \to x$

$\forall x \in X$. $\square$

(1.8) <u>Example</u> (diagonal semigroup).

Let $X = \ell^2$, $\lambda_n \in \mathbb{C}$, $\operatorname{Re} \lambda_n \leq \omega$.

Let $T(t)x = (e^{\lambda_n t} x_n)_{n\in\mathbb{N}}$.

Then T is a C_0-semigroup.

<u>Proof.</u> a) $$\|T(t)x\|^2 = \sum_{n=1}^{\infty} |e^{2\lambda_n t} x_n|^2$$

$$\leq e^{2\operatorname{Re}\lambda_n t} \sum |x_n|^2$$

$$\leq e^{2\omega t} \|x\|^2$$

Thus $\|T(t)\| \leq e^{\omega t}$.

b) $X_0 = c_{00} = \{ x \in \ell^2 : \exists n_0 \quad x_n = 0 \text{ if } n \geq n_0 \}$

finitely supporte sequences

c_{00} is dense in X.

$x \in c_{00}$ $\cancel{x}$ $(T(t)x)_k = e^{t\lambda_k} x_k \to x_k$

$t \to 0$ $\Rightarrow$ $T(t)x \to x$ $(t \downarrow 0)$ $\forall x \in c_{00}$.

Proposition 1.6 $\Rightarrow$ claim $\square$

<u>Rk</u> $T(t) \to I$ in $\mathcal{L}(\ell^2)$ as $t \downarrow 0$ (exercise).

iff $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$.

(1.7) <u>Example (Shift semigroup).</u>

Let ~~X~~ $X = L^p(\mathbb{R})$, $1 \le p < \infty$

$(T(t)f)(x) = f(x+t)$.

Then T is a C_0-semigroup.

<u>Proof.</u> a) $\|T(t)f\| = \|f\|$ $\forall f$

Thus $\|T(t)\| \le 1$.

b) Let $f \in C_c(\mathbb{R})$ (continuous vanishing outside a compact set). Then f is uniformly continuous; i.e. let $\varepsilon > 0$. Then $\exists\, \delta > 0$ $|f(x) - f(y)| \le \varepsilon$ if $|x-y| \le \delta$. Let $a > 0$ such that $f(x) = 0$ for $|x| > a$. Let $0 < t \le \delta$

$$\|(T(t)f)(x) - f(x)\|_p^p \le$$

$$\int_{\mathbb{R}} |f(x+t) - f(x)|^p = \int_{-a}^{a+1} \varepsilon^p \le (2a+1)\,\varepsilon^p$$

Thus $\|T(t)f - f\|_p \to 0$ $(t \downarrow 0)$.

Proposition 1.5 $\Rightarrow$ claim. □