§ 2 Der Generator einer C_0-Halbgruppe.

Let $T : [0,\infty) \to X$ be a C_0-semigroup. $T(0) = 0$.

(2.1) Definition. The generator A of T is defined by

$$D(A) := \{x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\}$$

$$Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t}.$$

Thus: $D(A) \subset X$ subspace (the domain of A).

$A : D(A) \to X$ is linear.

(2.2) **Proposition.** $x \in D(A) \Rightarrow$

$T(t)x \in D(A)$ & $AT(t)x = T(t)Ax$.

Proof. $\lim_{h \downarrow 0} \frac{1}{h}(T(h)T(t)x - T(t)x)$

$= T(t) \lim_{h \downarrow 0} \frac{1}{h}(T(h)x - x)$

$= T(t)Ax$. □

(2.3) **Cauchy problem.**

$$\begin{cases} \dot{u}(t) = Au(t) & t > 0 \\ u(0) = x \end{cases}$$

classical solution: $u \in C^1([0,\infty); X)$,

$u(t) \in D(A) \quad \forall t \geq 0$.

Theorem. $\forall x \in D(A)$ $\exists!$ u classical solution.

Proof. a) **Existence** $u(t) = T(t)x$

$$\frac{u(t+h) - u(t)}{h} = \frac{T(h)T(t)x - T(t)x}{h}$$

$= AT(t)x$.

b) <u>uniqueness.</u> Let u be a solution,

$t > 0$, $v(s) = T(t-s)u(s)$.

$\dot v(s) = -AT(t-s)u(s) + T(t-s)\dot u(s)$

$= -AT(t-s)u(s) + T(t-s)Au(s)$

$= 0$.

$\Rightarrow$ $v(0) = v(t)$ $\Rightarrow$ $T(t)x = u(t)$.

$\square$

(2.4) <u>Riemann integral.</u>

X Banach space

$u \in C([a,b]; X)$

$\|u\|_\infty := \max_{t\in[a,b]} \|u(t)\|$

π = <u>partition</u> $a = t_1 < \dots < t_n = b$
with <u>intermediate points</u> $s \in [t_j, t_{j+1}]$
(pip)

$|\pi| = \max |t_j - t_{j-1}|$

$S(\pi, u) = \sum_{j=1}^{n} u(s_j)(t_j - t_{j-1})$

<u>Theorem.</u> $\int_a^b u(t)\,dt := \lim_{|\pi|\to 0} S(\pi, u)$ exists in X.

(2.5) <u>Proposition.</u> $B \in \mathcal{L}(X, Y)$ + * $\Rightarrow$

$$B \int_a^b u(t)\,dt = \int_a^b Bu(t)\,dt$$

<u>Proof.</u> $B\,S(\pi, u) = S(\pi, Bu)$ $\square$

$X' = \{ x' : X \to \mathbb{K} : \text{ cont. linear} \} = \mathcal{L}(X, \mathbb{K})$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

<u>dual space</u>

<u>HB</u> $\|x\| = \sup_{\|x'\| \le 1} |\langle x', x\rangle|$

In particular: X' <u>separates</u> X;

(2.6) <u>Corollary.</u> $\langle x', \int_a^b u(t)\,dt \rangle = \int_a^b \langle x', u(t)\rangle\,dt$.

(2.7) <u>Corollary:</u> $\| \int_a^b u(t)\,dt \| \le \int_a^b \|u(t)\|\,dt$

<u>Proof.</u> $\| \int_a^b u(t)\,dt \| = \sup_{\|x'\| \le 1} |\langle x', \int_a^b u(t)\,dt\rangle|$

$$\leq \sup_{\|x'\| \leq 1} \int_a^b |\langle x', u(t)\rangle| \, dt$$

$$\leq \int_a^b \|u(t)\| \, dt . \quad \square$$

(2.6) <u>Fundamental Theorem.</u>

a) $u \in C([a,b];X)$, $v(t) = \int_a^t u(s)\,ds$

$\Rightarrow v \in C^1([a,b];X)$ & $v' = u$.

b) $u \in C^1([a,b];X)) \Rightarrow$

$$u(b) - u(a) = \int_a^b u'(s)\,ds .$$

<u>Proof of b)</u> $\langle x', u(b)\rangle - \langle x', u(a)\rangle$

$$= \int_a^b \frac{d}{dt} \langle x', u(t)\rangle \, dt$$

$$= \int_a^b \langle x', \dot u(t)\rangle \, dt$$

$$= \langle x', \int_a^b \dot u(t)\,dt \rangle$$

X' separates X $\Rightarrow$ claim. $\square$

T C_0-semigroup with generator A.

(2.7) <u>Proposition</u>. Let $x, y \in X$. Equivalent.

(i) $x \in D(A)$ & $Ax = y$;

(ii) $\int_0^t T(s) y \, ds = T(t)x - x$.

<u>Proof.</u> (ii) $\Rightarrow$ (i)

$$\frac{1}{t}(T(t)x - x) = \frac{1}{t}\int_0^t T(s) y \, ds$$
$$\longrightarrow T(0)y = y. \quad \text{as } t \downarrow 0.$$

Thus $x \in D(A)$ & $Ax = y$.

(i) $\Rightarrow$ (ii) $\quad u(t) = T(t)x$, $\quad x \in D(A)$

$$\Rightarrow \quad \dot{u}(t) = T(t)Ax$$
$$\Rightarrow \quad T(t)x - x = u(t) - u(0) = \int_0^t T(s) Ax \, ds$$

$\square$

(2.8) <u>Definition</u>. An operator A on X is <u>closed</u> if

$D(A) \ni x_n \to x$, $Ax_n \to y$ $\Rightarrow$ $x \in D(A)$ & $Ax = y$.

<u>Remark.</u> Let $D(A) = X$

A closed $\Leftrightarrow$ $A \in \mathcal{L}(X)$

(closed graph theorem).

(2.9) <u>Proposition.</u> The generator of a C_0-semigroup is closed.

<u>Proof.</u> Let $x_n \in D(A)$, $x_n \to x$,

$y_n := A x_n \to y$

$$\int_0^t T(s) y_n \, ds = T(t) x_n - x_n$$

$$\overset{n \to \infty}{\Longrightarrow} \int_0^t T(s) y \, ds = T(t) x - x$$

$\overset{(2.8)}{\Longrightarrow}$ $x \in D(A)$, $Ax = y$. □

(2.10) Lemma. Let $A : D(A) \to X$ be an operator. Then

$$\|x\|_A := \|Ax\| + \|x\|$$

defines a norm on $D(A)$. Equi

(i) A is closed;

(ii) $(D(A), \|\ \|_A)$ is complete.

(exercise)

(2.11) Proposition. Let $u \in C([a,b]; X)$ s.t. $u(t) \in D(A)$ and $Au \in C([a,b], X)$. Then ~~$u\in$~~ $\int_a^b u(t)\,dt \in D(A)$ and

$$A\int_a^b u(t)\,dt = \int_a^b Au(s)\,ds$$

Proof.

$$S(\pi_n, u) \in D(A)$$

$$S(\pi_n, u) \to \int_a^b u(t)\,dt$$

$$AS(\pi_n, u) \to y \qquad \square$$

(2.12) <u>Proposition</u>. (<u>Everyday formula</u>)

$x \in X \Rightarrow$

(AWF) $\int_0^t T(s)x\,ds \in D(A)$ & $A\int_0^t T(s)x\,ds = T(t)x - x$

German: <u>Allerweltsformel</u> = <u>AWF</u>.

<u>Proof</u>. $\frac{1}{h}\left[T(h)\int_0^t T(s)x\,ds - \int_0^t T(s)x\,ds\right]$

$$= \frac{1}{h}\left[\int_0^t T(s+h)x\,ds - \int_0^t T(s)x\,ds\right]$$

$$= \frac{1}{h}\left[\int_h^{t+h} T(r)x\,dr - \int_0^t T(r)x\,dr\right]$$

$$= \frac{1}{h}\left[\int_t^{t+h} T(r)x\,dr - \int_0^h T(r)x\,dr\right]$$

$$\longrightarrow T(t)x - x. \quad \square$$