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§ 3 The resolvent ~~of the generator~~ set of an operator.

X Banach space over $\mathbb{K}$.

Let A be an operator on X.

$$\varrho(A) := \{\lambda \in \mathbb{K} : \lambda - A : D(A) \to X \text{ bij. } \& \ (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

<u>resolvent</u> set. $R(\lambda, A) := (\lambda - A)^{-1}$

(3.1) <u>Remark.</u> a) $\varrho(A) \neq \emptyset \Rightarrow A$ closed

b) A closed $\Rightarrow$
$\varrho(A) = \{\lambda \in \mathbb{K} : \lambda - A$ bijective$\}$

~~(3.2) Theorem. $\varrho(A)$ is open~~

c) A, B operators
$A \subset B$, $\varrho(A) \cap \varrho(B) \neq \emptyset$
$\Rightarrow A = B$.

(3.2) <u>Resolvent identity.</u>

$$\frac{R(\lambda,A) - R(\mu,A)}{\mu-\lambda} = R(\lambda,A)\,R(\mu,A)$$

$\lambda,\mu \in \varrho(A)$ $\lambda \neq \mu$.

(3.3) <u>Proposition.</u> $\varrho(A)$ is open in $\mathbb{K}$.

More precisely, $\lambda_0 \in \varrho(A)$,

$$|\lambda-\lambda_0|\;\|R(\lambda_0,A)\| < 1 \Rightarrow \lambda \in \varrho(A)$$

$$\& \quad R(\lambda,A) = \sum_{n=0}^{\infty} (\lambda_0-\lambda)^n R(\lambda_0,A)^{n+1}$$

<u>Proof.</u> Assume $|\lambda-\lambda_0|\;\|R(\lambda_0,A)\| = q < 1$

$$(\lambda - A) = (\lambda-\lambda_0 + \lambda_0 - A)$$

$$= (I - (\lambda_0-\lambda)R(\lambda_0,A))\,\cancel{R(\lambda_0,A)}\,(\lambda_0-A)$$

$$\Rightarrow \quad R(\lambda,A) = R(\lambda_0,A)\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n R(\lambda_0,A)^n.$$

In fact, $S := (\lambda_0-\lambda)\,R(\lambda_0,A) \in \mathcal{L}(X)$,

$\|S\| < 1$. $\Rightarrow$ $I-S$ invertible.

$$R(\lambda, A) = R(\lambda_0, A)(I-S)^{-1}.$$

Proof. Rf $(\lambda - A) R(\lambda_0, A)(I-S)^{-1}x =$

$(\lambda - \lambda_0 + \lambda_0 - A) R(\lambda_0, A)(I-S)^{-1}x =$

$((\lambda - \lambda_0) R(\lambda_0, A) + I)(I-S)^{-1}x = x.$

$\forall x \in X.$

$x \in D(A) \Rightarrow R(\lambda_0, A)(I-S)^{-1}(\lambda - A)x$

$= (I-S)^{-1} R(\lambda_0, A)(\lambda - A)x$

$= (I-S)^{-1} R(\lambda_0, A)(\lambda - \lambda_0 + \lambda_0 - A)x$

$= (I-S)^{-1}(I-S)x = x.$ □

(3.4) Corollary. Let $\lambda_0 \in \mathbb{K}$.
$\exists \lambda_n \in \rho(A)$, $\lambda_n \to \lambda_0$ &
$|\lambda_n| \leq c \Rightarrow \lambda_0 \in \rho(A).$

~~Proof.~~ ~~dist $(\lambda, \sigma$~~

(3.5) Corollary. Let $\lambda \in \rho(A)$.
Then $\operatorname{dist}(\lambda, \sigma(A)) \geq \|R(\lambda, A)\|^{-1}$

Proof of (3.5). Let $\mu \in \sigma(A)$. Then
$|\lambda - \mu| \, \|R(\lambda, A)\| \geq 1 \Rightarrow (\lambda - \mu) \geq \|R(\lambda, A)\|^{-1}$
$\forall \mu \in \sigma(A)$. □

Here $\sigma(A) := \mathbb{K} \setminus \rho(A)$ is the spectrum of A.

Proof of (3.4). ~~dist (λ_n, σ~~
$|\lambda_n - \lambda_0| \, \|R(\lambda_n, A)\| \le c\,|\lambda_n - \lambda_0| < 1$ if n is big enough. Thus $\lambda_0 \in \rho(A)$. □

(3.6) <u>Yosida approximation.</u> Let A be an operator such that $(\omega, \infty) \subset \rho(A)$ & $\|\lambda R(\lambda, A)\| \le M \quad (\lambda > \omega)$.

~~F~~ Then

$$\overline{D(A)} = X \quad \text{iff} \quad \lambda R(\lambda, A)x \to x \quad (\lambda \to \infty) \quad \forall x \in X.$$

<u>Pf.</u> „$\Leftarrow$" trivial

„$\Rightarrow$" 1. $x \in D(A)$ $\quad x = \lambda R(\lambda, A)x - R(\lambda, A)Ax$

$\Rightarrow \quad \lambda R(\lambda, A)x - x = R(\lambda, A)Ax \to 0$ $(\lambda \to \infty)$.

2. Equicontinuity Lemma. □

Assume $\quad \| \lambda R(\lambda, A) \| \leq M \qquad (\lambda > \omega)$

and $\quad \overline{D(A)} = X$.

Define: $\quad A_n = n^2 R(n, A) - n \in \mathcal{L}(X)$.

Then $\quad A_n x \longrightarrow Ax \quad$ for all $x \in D(A)$

<u>Proof.</u> $\quad n R(n, A) x \longrightarrow x \quad n \to \infty$

$$n R(n, A) x - R(n, A) A x = x$$

$$\Rightarrow \quad R(n, A) A x = n R(n, A) x - x$$

$$\Rightarrow \quad n R(n, A) A x = n^2 R(n, A) x - n x$$

$$\longrightarrow Ax \quad (n \to \infty). \quad \square$$

§ 4 The resolvent of the generator.

T C_0-sg, A generator. $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

(4.1) **Proposition.** (rescaling)

a) $\lambda \in \mathbb{K} \Rightarrow S(t) = e^{\lambda t} T(t)$ defines a C_0-sg. Generator: $B = A + \lambda$, $D(B) = D(A)$.

b) $\alpha > 0$ $S(t) = T(\alpha t)$ C_0-sg generator $B = \alpha A$.

(4.2) **Corollary.** a) $\lambda \in \mathbb{K}$, $x \in X \Rightarrow$ $\int_0^t e^{-\lambda s} T(s) x \, ds \in D(A)$ &

$$(A-\lambda) \int_0^t e^{-\lambda s} T(s) x \, ds = e^{-\lambda t} T(t) x - x$$

b) $\lambda \in \mathbb{K}$, $x \in D(A) \Rightarrow$

$$\int_0^t e^{-\lambda s} T(s) (\lambda - A) x \, ds = e^{-\lambda t} T(t) x - x$$

$\|T(t)\| \le M e^{\omega t}$

(4.3) <u>Proposition.</u> Let $\lambda \in \mathbb{K}$, $\mathrm{Re}\lambda > \omega$.
Then $\lambda \in \varrho(A)$ &

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \qquad (1)$$

$\forall x \in X$.

Proof. $Q_t x = \int_0^t e^{-\lambda s} T(s)x \, ds$

~~$\in \mathcal{L}(x$~~ $Q_t \in \mathcal{L}(X)$.

Rk. $\|e^{-\lambda t} T(t)x\| \le M e^{(-\mathrm{Re}\lambda + \omega)t} \|x\|$
$\Rightarrow$ (1) converges and defines

$Q \in \mathcal{L}(X)$.

Moreover $Q_t x \longrightarrow Qx \quad t \to \infty \quad \forall x$

$Q_t x \in D(A)$ & $(\lambda - A)Q_t x = x - e^{-\lambda t} T(t)x$
$\longrightarrow x \quad (t \to \infty)$.

$\lambda - A$ closed $\Rightarrow Qx \in D(A)$ &

$(\lambda - A)Qx = x$. If $x \in D(A)$ then

$Q(\lambda - A)x = \cancel{(\lambda - A)Qx} \lim Q_t(\lambda - A)x$
$= \cancel{(\lambda - A)} \lim (\lambda - A)Q_t x = x$

$\square$

<u>Consequence</u> : $\| R(\lambda, A) \| \leq \frac{M}{\mathrm{Re}\,\lambda - \omega}$.

(4.4) <u>Corollary</u>. $\| T(t) \| \leq 1$

(i.e. contraction semigroup)

$\Rightarrow \; (0, \infty) \subset \rho(A) \;\&\; \| \lambda R(\lambda, A) \| \leq 1$

$(\lambda > 0)$.

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