§ 5 The Hille-Yosida Theorem.

T contractive $:\Leftrightarrow$ $\|T(t)\| \leq 1$

contraction semigroup.

(5.1) **Theorem** (Hille-Yosida)

Let A be an operator on X

Equivalent.

(i) A generates a contractive C_0-semigroup.

(ii) (a) $\overline{D(A)} = X$

(b) $(0,\infty) \subset \varrho(A)$ &

$\|\lambda R(\lambda, A)\| \leq 1 \quad (\lambda > 0)$.

Proof. (i) $\Rightarrow$ (ii) done

(ii) $\Rightarrow$ (i) $\quad A_n = n^2 R(n,A) - nI$

$$e^{tA_n} = e^{-ntI} e^{tn^2 R(n,A)}$$

$$\Rightarrow \quad \|e^{tA_n}\| \leq 1 \qquad (t>0,\ n \in \mathbb{N}).$$

$$e^{tA_n}x - e^{tA_m}x =$$

$$\int_0^t \frac{d}{ds}\left(e^{(t-s)A_n} e^{sA_m}x\right) ds =$$

$$\int_0^t e^{(t-s)A_n} (A_m - A_n) e^{sA_m} x \, ds =$$

$$\int_0^t e^{(t-s)A_n} e^{sA_m} (A_m - A_n) x \, ds$$

$$\Rightarrow \|e^{tA_n}x - e^{tA_m}x\| \leq t \, \|(A_m - A_n)x\|$$

Let $x \in D(A)$. Then $A_n x \to Ax$

(Yosida approximation).

Thus $e^{tA_n}x$ Cauchy for $x \in D(A)$

Equicontinuity Lemma $\Rightarrow$

Define $F_n^\tau : X \longrightarrow C([0,\tau],X)$

by $(F_n^\tau x)(t) = e^{tA_n}x$

F_n^τ is linear $\|F_n^\tau x(t)\| \leq \|x\| \quad \forall t$

$\Rightarrow \|F_n^\tau x\|_\infty \leq \|x\| \quad \Rightarrow \|F_n^\tau\| \leq 1.$

$\|(F_n^\tau - F_m^\tau)(x)\|_\infty \leq \tau \, \|Ax_n - Ax_m\|$

if $x \in D(A)$

Thus $(F_n^\tau x)_{n\in\mathbb{N}}$ converges in $C([0,\tau];X)$

whenever $x \in D(A)$.

Equicontinuity Lemma $\Rightarrow$ convergence

in $C([0,\tau],X) \quad \forall x \in X$.

Thus $\exists\; T(.) : \mathbb{R}_+ \to \mathcal{L}(X)$ strongly

continuous

$$T(t)x = \lim_{n\to\infty} e^{tA_n}x$$

$\forall$ unif^{ly} on $[0,\tau] \quad \forall x \in X$.

$$T(t)\,T(s)x \overset{(1)}{=} \lim_{n\to\infty} e^{tA_n} e^{sA_n} x$$

$$= \lim e^{(t+s)A_n} x$$

$$= T(t+s)\,x$$

(1) $y_n = e^{sA_n}x \longrightarrow \cancel{e^s}\, T(s)x$

$\Rightarrow e^{tA_n} y_n \longrightarrow \cancel{e^t}\, T(t)T(s)\,x$.

~~Let~~ Thus T is a C_0-sg.

Let B be the generator of T.

Let $x \in D(A)$, $Ax = y$.

$$\int_0^t e^{sA_n} A_n x \, ds = e^{tA_n}x - x$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\int_0^t \cancel{e}\,T(s)y \, ds \qquad\qquad T(t)x - x$$

since ~~deriv~~ $e^{sA_n} A_n x \longrightarrow T(s)Ax$ uniformly ~~glob~~

on ~~[...]~~ $[0,t]$. ($T_n^t x \longrightarrow T(.)x$ in

$C([0,t], X)$ $\forall x \Rightarrow T_n^t A_n x \longrightarrow T(.)Ax$
in $C([0,t], X)$.)

Char. of the generator $\Rightarrow$ $x \in D(B)$

& $Bx = Ax$.

Thus $A \subset B$. Hence $A = B$. $\square$

Let T be a bounded C_0-sg.

$\|T(t)\| \leq M$.

$$\|x\|_0 := \sup_{s \geq 0} \|T(s)x\| .$$

equivalent norm.

$\|T(t)x\|_0 \leq \|x\|_0$; i.e. $\|T(t)\|_0 \leq 1$.

A generator. $\Rightarrow$ $\|\lambda R(\lambda, A)\|_0 \leq 1$.

$\Rightarrow$ $\|[\lambda R(\lambda, A)]^n x\| \leq \|[\lambda R(\lambda, A)]^n x\|_0$

$\leq \|x\|_0 \leq M\|x\|$.

(5.2) <u>Corollary</u>. Let A be an operator. $M \geq 1$. Equi

(i) A generates a C_0-sg T s.t. $\|T(t)\| \leq M$ $(t \geq 0)$;

(ii) (a) $\overline{D(A)} = X$;

(b) $(0,\infty) \subset \rho(A)$;

(c) $\|(\lambda R(\lambda, A))^n\| \leq M$ $\forall n \in \mathbb{N}_0$, $\lambda > 0$.

<u>Proof</u>. (i) $\Rightarrow$ (ii) done

(ii) $\Rightarrow$ (i) $A_n = n^2 R(n, A) - n$.

$$e^{t n^2 R(n,A)} = \sum_{k=0}^{\infty} \frac{t^k n^k}{k!} [n R(n,A)]^k$$

$$\|e^{t n^2 R(n,A)}\| \leq \sum_{k=0}^{\infty} \frac{t^k n^k}{k!} M \leq M e^{tn}$$

$\Rightarrow$ $\|e^{tA_n}\| \leq M$.

Now the proof of (5.1) goes through. □

§ 6 Groups.

(6.1) Definition. A mapping $T : \mathbb{R} \to \mathcal{L}(X)$ is a group if

$$T(t+s) = T(t)\,T(s) \qquad (t, s \in \mathbb{R})$$

$$T(0) = I.$$

and a C_0-semigroup if in addition

$$\lim_{t \downarrow 0} T(t)x = x \qquad \forall x \in X.$$

(6.2) Consequences a) $T(t)^{-1} = T(-t)$. bded [1]

b) $(T(t))_{t \geq 0}$ is a C_0-sg, hence

$$\|T(t)\| \leq M e^{wt} \qquad t \geq 0$$

c) $T : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous

1) Theorem of the continuous inverse.

Proof. Let $t_n \to t$. Choose $s > 0$ such that $t_n + s \geq 1$. $\Rightarrow$

$$T(t_n)x - T(t)x = T(-s)\left[T(s+t_n)x - T(s+t)x\right]$$

$$\to 0 \quad (n \to \infty) \quad \square$$

d) ~~$(T(-t))_{t \geq 0}$ is a C_0-semigroup.~~

~~Its generator is A.~~

$t \in \mathbb{R} \Rightarrow T(t)D(A) \subset D(A)$ &

$AT(t)x = T(t)Ax \quad (x \in D(A))$

Pf. $$\frac{T(h)T(t)x - T(t)x}{h} = T(t)\frac{T(h)x - x}{h} \to T(t)Ax$$

$h \downarrow 0$

e) $x \in D(A) \Rightarrow \frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$

$\forall x \in D(A)$.

Proof. $$\frac{T(t-h)x - T(t)x}{-h} = T(-s)\frac{T(t+s-h) - T(t+s)x}{-h}$$

$$\to T(-s)AT(t+s)x = T(-s)AT(t+s)x$$

$$\overset{d)}{=} AT(t)x$$

property of C_0-semigroups.

$s + t > 0$

$\square$

f) $(T(-t))_{t\geq 0}$ is a C_0-semigroup and

$-A$ its generator.

Pf. C_0-sg : clear.

Let B be the generator of $(T(-t))_{t\geq 0}$,

Let $x \in D(A) \overset{e)}{\Rightarrow}$

$$\lim_{t\downarrow 0} \frac{T(-t)x - x}{-t} = Ax$$

$$\Rightarrow \lim_{t\downarrow 0} \frac{T(-t)x - x}{t} = -Ax$$

$\Rightarrow -A \subset B$. ~~$\Rightarrow A = B$. If $x \in D(B)$~~

g) For each $x \in D(A)$ $\exists!$ solution of

$$\begin{cases} u \in C^1(\mathbb{R}, X), \quad u(t) \in D(A) & (t \in \mathbb{R}) \\ \dot{u}(t) = Au(t) & (t \in \mathbb{R}) \\ u(0) = x & \end{cases}$$

Namely : $T(t)x = u(t)$

Pf. existence : e)

uniqueness : from sg case. □

h) $\quad \|T(t)\| \le M e^{w|t|}$ for some $M \geq 0$, $w \in \mathbb{R}_+$

Pf. clear for $t \geq 0$ $\quad \|T(t)\| \le M_+ e^{\omega_+ t}$ $\quad t \geq 0$

$\|T(-t)\| \le M_- e^{-\omega_- t}$ $\quad t < 0$

$= M_- e^{\omega_- |t|}$ □

i) $\quad \sigma(A) = \{ \lambda \in \mathbb{C} : \cancel{|\lambda|}\ \mathrm{Re}\,\lambda \le \omega \}$

ω as in (h) and $\quad \|R(\lambda, A)\| \le \dfrac{M}{|\mathrm{Re}\,\lambda| - \omega}$

Proof. $\mathrm{Re}\,\lambda < -\omega \Rightarrow \cancel{(\lambda + A)^{-1}}$ ~~ex.~~

$\Rightarrow \omega < -\mathrm{Re}\,\lambda \Rightarrow -\lambda \in \varrho(-A) \Rightarrow$

$\lambda \in \varrho(A)$. □

41.1

Missing ~~implis~~ inclusion $B \subset -A$

in f)

Let $x \in D(B)$ $\quad \frac{T(-t)x - x}{t} \to Bx = y$.

Let $\lambda \in \varrho(A)$. $\overset{\text{via}}{\Rightarrow}$ $\frac{T(-t)R(\lambda,A)x - R(\lambda,A)x}{t}$

$\to R(\lambda,A)y$ $\quad t \downarrow 0$

d) $\Rightarrow$ $-AR(\lambda,A)x = R(\lambda,A)y$

$\lambda R(\lambda,A)x - AR(\lambda,A)x = x$

$\Rightarrow R(\lambda,A)y = x - \lambda R(\lambda,A)x$ $\quad \Rightarrow x \in D(A)$.

$\square$