<u>Lemma.</u> $T$ Halbgruppe : $(0,\infty) \to \mathcal{L}(X)$

$t_0 > 0$.

a) $T(t_0)$ inj. $\Rightarrow$ $T(t)$ inj. $\forall t>0$

b) $T(t_0)$ surj. $\Rightarrow$ $T(t)$ surj. $\forall t>0$

<u>Pf.</u> a) 1. Let $0 < t < t_0$, $T(t)x = 0$

$\Rightarrow$ $0 = T(t_0-t)\,T(t)x = T(t_0)x \Rightarrow x = 0$.

2. $T(nt_0)$ inj. $T(t_0)^n x = 0 \Rightarrow T(t_0)^{n-1}x = 0$

$\ldots \Rightarrow T(t_0)x = 0 \Rightarrow x = 0$

3. Let $0 < t$ be arbitrary. Choose $nt_0 > t$. 1. & 2. $\Rightarrow$ $T(t)$ inj.

b) 1. Let $0 < t < t_0$. Let $y \in X$.
$\exists x \in X$ $T(t_0)x = y$ $\Rightarrow$ $T(t)\,T(t_0-t)x = y$

2. $T(nt_0)$ surj.

3. Let $t > 0$ be arbitrary. $\exists n$ $nt_0 > t$

1.82. $\Rightarrow$ $T(t)$ surj. $\square$

(6.3) Theorem. Let $T$ be a $C_0$-sg with generator $A$. Equ.

(i) $\exists\, \mathcal{U}$ a $C_0$-group s.t. $T(t) = \mathcal{U}(t)$ $(t \geq 0)$

(ii) $\exists\, t_0 > 0$ $T(t_0)$ is bijective;

(iii) $-A$ generates a $C_0$-sg.

Proof. (ii) $\Rightarrow$ (i) $\mathcal{U}(t) := T(t)$ for $t \geq 0$, (cf. Lemma).
$\mathcal{U}(-t) := T(t)^{-1}$ for $t \geq 0$.

Claim: $\mathcal{U}(t_1 + t_2) = \mathcal{U}(t_1)\,\mathcal{U}(t_2)$ $\forall\, t_1, t_2 \in \mathbb{R}$

Clear if $t_1 > 0, t_2 > 0$ or $t_1 < 0, t_2 < 0$.

a) Assume $t_1 + t_2 > 0$, $t_1 < 0$

Then $t_2 > 0$ and $\mathcal{U}(t_2) =$
$T(t_2 - (t_1 + t_2))\, T(t_1 + t_2) = T(-t_1)\, T(t_1 + t_2)$
$\Rightarrow \mathcal{U}(t_1 + t_2) = \mathcal{U}(t_1 + t_2) = T(t_1)\, \mathcal{U}(t_2)$
$= \mathcal{U}(t_1)\, \mathcal{U}(t_2)$.

b) Assume $t_1 + t_2 < 0$, $t_2 > 0$. Then
$\mathcal{U}(t_1 + t_2) = \mathcal{U}(-t_1 - t_2)^{-1} \overset{a)}{=} (\mathcal{U}(-t_1)\,\mathcal{U}(-t_2))^{-1} = \mathcal{U}(t_2)\,\mathcal{U}(t_1)$

By our definition $\mathcal{U}$ is a $C_0$-group. ~~and~~

$\cancel{A}$

(i) $\Rightarrow$ (iii) see consequences.

(iii) $\Rightarrow$ (ii). Denote by $S$ the $C_0$-sg generated by $-A$. Let $x \in D(A)$.

Then $\frac{d}{dt} T(t) S(t)x = A T(t) S(t) x + T(t)(-A S(t)x)$

$= 0$. Thus $T(t) S(t) x \equiv \text{const} = T(0) S(0) x = x$

Thus $T(t) S(t) = I$. Similarly $S(t) T(t) = I$

$\Rightarrow$ (i). $\square$

As application of the HY we show the following.

(6.4) Theorem. Let $A$ be the generator of an isometric $C_0$-group. Then $A^2$ generates a contractive $C_0$-semigroup.

$$D(A^2) := \{ x \in D(A) : Ax \in D(A) \}$$

Proof. We know that $(0,\infty) \subset \rho(\pm A)$ & $\|\lambda R(\lambda, A)\| \le 1$, $\|\lambda R(\lambda, -A)\| \le 1$

$$R(\lambda, -A) = (\lambda + A)^{-1}.$$

Let $x \in D(A^2)$, $\lambda > 0$

$(\lambda^2 - A^2)x = (\lambda - A)(\lambda + A)x$.

$\Rightarrow \lambda^2 \in \rho(A)$ & $R(\lambda^2, A^2) = R(\lambda, -A)\, R(\lambda, A)$.

In fact, $R(\lambda, -A)\, R(\lambda, A)(\lambda^2 - A^2)\, x = x$

$(x \in D(A^2))$.

Conversely, let $y \in X$, $x = R(\lambda, -A)\, R(\lambda, A)y$

$\Rightarrow$ $x \in D(A)$ & $\lambda x + Ax = R(\lambda, A)y$

$\Rightarrow$ $x \in D(A^2)$ & $\underbrace{(\lambda - A)(\lambda + A)x}_{(\lambda^2 - A^2)x} = y$

$\| \lambda^2 R(\lambda^2, A^2) \| \leq \| \lambda R(\lambda, -A) \| \, \| \lambda R(\lambda, A) \| \leq 1$.

Let $y \in X$ $\quad \lambda^2 R(\lambda^2, A^2) y =$

$\lambda R(\lambda, -A)\, \lambda R(\lambda, A) y \to y$ $\quad \lambda \to \infty$ $\quad \square$

Rq: $\quad S_n x \to Sx \quad \forall x \in X$

$x_n \to x \Rightarrow S_n x_n \to Sx$.

Pf. $\quad S_n x_n - Sx = S_n(x_n - x) + S_n x - Sx$

$\to 0 \quad \square$

Example. $X = L^p(\mathbb{R})$, $1 \leq p < \infty$

$(S(t)f)(x) = f(x+t)$ isometric $C_0$-group

generator $B$

$D(B) = \cancel{\{f \in} \, W^{1,p}(\mathbb{R})$

Thus

$D(B^2) = W^{2,p}(\mathbb{R})$ $B^2 f = f''$.

Let $T$ be the sg generated by

$B^2$.

ÜA $(T(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4t} f(y)\,dy$.

ÜA

1. Sei $A$ ein Operator mit $\varrho(A) \neq \emptyset$, $S \in \mathcal{L}(X)$. Äqu.

(i) $\exists \lambda \in \varrho(A) \quad R(\lambda, A) S = S R(\lambda, A)$

(ii) $x \in D(A) \Rightarrow Sx \in D(A) \,\&\, ASx = SAx$

(iii) $\forall \lambda \in \varrho(A) \quad R(\lambda, A) S = S R(\lambda, A)$.

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