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§ 7 Dissipative operators.

We start by some operator theory.

(7.1) Lemma (a priori estimate).

Let $\Lambda \subset \mathbb{C}$ be open, connected, $M : \Lambda \to (0,\infty)$ continuous. A an operator such that

(a) $\| \lambda x - Ax \| \geq M(\lambda) \|x\|$ $(\lambda \in \Lambda,\ x \in D(A))$

(b) $\exists \lambda_0 \in \Lambda$ $(\lambda_0 - A)$ surjective.

Then $\Lambda \subset \rho(A)$ & $\|R(\lambda, A)\| \leq \frac{1}{M(\lambda)}$ for all $\lambda \in \Lambda$.

Pf. 1. $\Lambda_0 := \rho(A) \cap \Lambda$ is open in Λ

2. $\lambda \in \Lambda_0 \Rightarrow \|R(\lambda, A)\| \leq \frac{1}{M(\lambda)}$.

3. Λ_0 is closed in Λ. Let $\lambda_n \in \Lambda_0$, $\lambda_n \to \mu$, $\mu \in \Lambda$. Then $\|R(\lambda_n, A)\| \leq \frac{1}{M(\lambda_n)}$

Since $\lambda_n \to \mu$, $M(\lambda_n) \longrightarrow M(\mu) > 0$

$\Rightarrow$ $\sup \| R(\lambda_n, A) \| < \infty$

$\Rightarrow$ $\mu \in \varrho(A)$.

4. $\Lambda_0 \neq \emptyset$ since $\lambda_0 \in \Lambda_0$. $\square$

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(7.2) <u>Definition</u>. An operator A is <u>dissipative</u> if

$$\| x - tAx \| \geq \| x \| \quad \forall t > 0, x \in D(A)$$

$$\left[\Leftarrow \quad \| \lambda x - Ax \| \geq \lambda \| x \| \quad \forall \lambda > 0, x \in D(A) \right]$$

A is <u>m-dissipative</u> if in addition

~~(7.3)~~ $\exists \lambda_0 > 0$ s.t. $\lambda_0 - A$ is surjective.

(7.3) <u>Theorem</u> (Lumer-Phillips). Let A be an operator. Equ.

(i) A generates a contraction C_0-sg

(ii) A is m-diss. & dd.

Since $\lambda_n \to \mu$, $M(\lambda_n) \to M(\mu) > 0$,

it follows that $\sup_{n \in \mathbb{N}} \| R(\lambda_n, A) \| < \infty$

$\Rightarrow \mu \in \varrho(A)$.

4. $\Lambda_0 \neq \emptyset$ since $\lambda_0 \in \Lambda_0$ $\square$

(7.2) Definition. An operator A is dissapative

if $\|x - tAx\| \geq \|x\|$ $\forall t > 0$, $x \in D(A)$

[$\Leftrightarrow$ $\|\lambda x - Ax\| \geq \lambda \|x\|$ $\forall \lambda > 0$, $x \in D(A)$],

A is m-dissipative if in addition

$\exists \lambda_0 > 0$ s.t. $(\lambda_0 - A) D(A) = X$

range condition.

(7.3) Theorem (Lumer - Phillips).

Let A be an operator. Equ:

(i) A generates a contractive C_0-sg

(ii) A is cld & ~~diss~~ m-dissipative.

(7.4) **Rk.** A m-diss $\iff$ $(0,\infty) \subset \varrho(A)$

& $\|\lambda R(\lambda, A)\| \leq 1$

follows from (7.1)

(7.4) & HY $\Rightarrow$ (7.3).

(7.5) Hilbert space:

Proposition. $X = H$. Equ.:

(i) A is dissipative

(ii) $\operatorname{Re}(Ax \mid x) \leq 0 \quad \forall x \in D(A)$.

Proof. (ii) $\Rightarrow$ (i) $\|x - tAx\|^2 =$

$(x - tAx \mid x - tAx) = \|x\|^2 - t\,2\operatorname{Re}(x \mid Ax) + t^2\|Ax\|^2$

$\geq \|x\|^2$

(i) $\Rightarrow$ (ii) $\quad -t\,2\operatorname{Re}(x \mid Ax) + t^2\|Ax\|^2 \geq 0 \qquad (t > 0)$

$\Rightarrow -2\operatorname{Re}(x \mid Ax) + t\|Ax\|^2 \geq 0 \qquad t \downarrow 0$

$\Rightarrow$ claim. $\square$

Weak convergence

(7.6) Recall. X Banach space

a) $x_n \rightharpoonup x \iff \langle x', x_n\rangle \longrightarrow \langle x', x\rangle$
$\forall x' \in X'$.

(weak convergence)

b) $S \in \mathcal{L}(X)$.

$\exists!\ S' \in \mathcal{L}(X')$ $\quad \langle Sx, x'\rangle = \langle x, S'x'\rangle$

c) $x_n \rightharpoonup x \quad \Rightarrow \quad Sx_n \rightharpoonup Sx$

Proof. $\langle Sx_n, x'\rangle = \langle x_n, S'x'\rangle$
$\longrightarrow \langle x, S'x'\rangle = \langle Sx, x'\rangle$. $\square$

d) $x_n \to x \quad \Rightarrow \quad x_n \rightharpoonup x \quad \Rightarrow \quad \sup_{n\in\mathbb{N}} \|x_n\| < \infty$

e) X reflexive $\iff$ $\big[\, \|x_n\| \le c \Rightarrow \exists\, x \in X$
& ss $x_{n_k} \rightharpoonup x \,\big]$

Examples: $L^p(\Omega)$ reflexive $1 < p < \infty$.
$L^1(\Omega)$ is not refl. $\emptyset \neq \Omega \subset \mathbb{R}^d$ open
$C(K)$ is not refl. if $\# K = \infty$.
each Hilbert space is reflexive

(7.7) <u>Proposition.</u> X reflexive

$(\omega, \infty) \subset \varrho(A)$

$\| \lambda R(\lambda, A) \| \leq M \qquad (\lambda > \omega)$

$\Rightarrow \quad \overline{D(A)} = X$

<u>Proof.</u> Let $x \in X$.

$\exists \; \lambda_n \to \infty \qquad \lambda_n R(\lambda_n, A) x \longrightarrow y$

Let $\mu \in \varrho(A) \quad \overset{c)}{\Rightarrow}$

$$\lambda_n R(\mu, A) R(\lambda_n, A) x \longrightarrow R(\mu, A) y$$

$$\|$$

$$\underbrace{\frac{\lambda_n}{\lambda_n - \mu} R(\mu, A) x}_{\downarrow \; R(\mu, A)x} - \underbrace{\frac{\lambda_n}{\lambda_n - \mu} \cdot R(\lambda_n, A) x}_{\downarrow \; 0}$$

$\Rightarrow \quad R(\mu, A) x = R(\mu, A) y$

$\Rightarrow \quad x = y.$

Thus $\quad x \in \overline{\cancel{D(A)}}^{w} \neq \overline{D(A)}$. $\boxtimes$

See below. $\square$

Recall. $C \subset X$ convex.

$x_n \in C$, $x_n \to x$ $\Rightarrow$ $x \in \overline{C}$.

(Hahn-Banach).

(7.8) Theorem. H Hilbert space.

A an operator on H. Equ.

(i) A generates a contractive C_0-sg

(ii) A is m-dissipative

(Lumer-Phillips).

Remark. a) H Hilbert space.

A dissipative. Then

A ~~d~~ m-dissipative $\Leftrightarrow$ A maximal dissipative

$! \Leftrightarrow [A \subset B$ B dissipative $\Rightarrow A = B]$

b) false in Banach spaces.

Let $x \in X$, ~~$x \neq 0$~~. Then $\exists x' \in X'$

$\|x'\| \leq 1$, $\langle x_\varepsilon', x \rangle = \|x\|$ (HB).

$J(x) := \{ x' \in X' : \|x'\| \leq 1, \langle x', x \rangle = \|x\| \}$.

(7.9) Proposition. ~~Ea~~ A operator. Equi:

(i) A is diss.

(ii) $\forall x \in D(A)$ $\exists x' \in J(x)$

$\mathrm{Re} \langle x', Ax \rangle \leq 0$.

Rk. $X = H$ Hilbert.

$x \neq 0 \Rightarrow J(x) = \{ \frac{x}{\|x\|} \}$

$\langle x \mid \frac{x}{\|x\|} \rangle = \frac{\|x\|^2}{\|x\|} = \|x\|$. □

Proof. Only (ii) $\Rightarrow$ (i)

~~$\|x\|$~~. Let $x \in D(A)$, $x' \in J(x)$ s.t.

$\mathrm{Re} \langle x', Ax \rangle \leq 0$

$\|x\| = \langle x', x \rangle \leq \mathrm{Re} \langle x', x - tAx \rangle$

$\leq \|x - tAx\|$. □

(7.10) <u>Proposition.</u> Let A be the generator of a contractive C_0-sg. Then A is <u>strictly dissipative</u>; i.e.

$\forall x \in D(A) \quad \forall x' \in J(x)$

$$\operatorname{Re} \langle x', Ax \rangle \leq 0 .$$

<u>Rk.</u> more is true ~~remains~~: A diss & dd $\Rightarrow$ strictly diss. (without proof)

<u>Pf of (7.10).</u> Let $x \in D(A)$, $x' \in J(x)$

$$\operatorname{Re}\langle Ax, x' \rangle = \lim_{t \downarrow 0} \left\langle \frac{T(t)x - x}{t}, x' \right\rangle$$

$$= \lim_{t \downarrow 0} \frac{1}{t}\Big[\langle T(t)x, x'\rangle - \langle x, x'\rangle\Big]$$

$$\leq \overline{\lim_{t \downarrow 0}} \frac{1}{t}\big(\|T(t)x\| - \|x\|\big) \leq 0 . \qquad \square$$

<u>Conclusion.</u>

(7.11) <u>Theorem.</u> Let A be dd. Equ.:

(i) A generates a contractive C_0-sg.

(ii) a) $\forall x \in D(A) \quad \exists x' \in \cancel{D(} J(x) \quad \operatorname{Re}\langle Ax, x'\rangle \leq 0$

b) $\exists \lambda_0 > 0 \quad (\lambda_0 - A)D(A) = X$.

(7.12) <u>Closable operators</u>.

<u>Rk.</u> Let $G \subset X \times X$

$\exists$ an operator A such that $G = G(A)$

$\Leftrightarrow$ $(0, y) \in G \Rightarrow y = 0$.

<u>Proposition.</u> Let A be an operator. Equ:

(i) $\exists$ an operator $\bar{A}$ s.t. $G(\bar{A}) = \overline{G(A)}$

(ii) $x_n \in D(A)$, $x_n \to 0$, $Ax_n \to y$ $\Rightarrow y = 0$.

In that $\bar{A}$ is ~~st~~ called <u>the closure of A</u>.

<u>Clear:</u>

$$D(\bar{A}) = \{ x \in X : \exists\, x_n \in D(A),\ x_n \to x,\ (Ax_n) \text{ converges} \}$$

$$\bar{A}x = \lim_{n \to \infty} Ax_n .$$

(7.13) <u>Proposition.</u> Let A be dissipative and ~~clos~~ dd. Then A is closable and $\bar{A}$ is dissipative

<u>Proof.</u> $x_n \to 0 \qquad Ax_n \to y$.

Let $z \in D(A)$. Then

$\| (x_n + tz) \cancel{+} t A(x_n + tz) \| \geq \| x_n + tz \| \qquad t > 0$

$\| x_n + tz - tAx_n - t^2 z \quad \| \geq \| x_n + tz \|$

$n \to \infty \quad \Rightarrow \quad t \| z - \overset{y}{\cancel{Ax_n}} - tz \| \geq t \|z\|$

$\Rightarrow \quad \| z - \overset{y}{\cancel{Ax_n}} - tz \| \geq \|z\| \qquad t \downarrow 0$

$\Rightarrow \quad \| z - y \| \geq \|z\|$

$z \to y \quad \Rightarrow \quad \|y\| \leq 0 \qquad \square$.

Let $x \in D(\bar{A})$, $\bar{A}x = y$.

$\Rightarrow \exists x_n \in D(A) \quad x_n \to x, \quad Ax_n \to y = \bar{A}x$

$\| x_n \cancel{+} - tAx_n \| \geq \|x_n\| \qquad n \to \infty$

$\| x - t\bar{A}x \| \geq \|x\|$. $\qquad \square$

(7.14) <u>Theorem (Lumer–Phillips).</u>

Let A be dissipative and d.d.

~~Assume~~ $\exists \lambda_0 > 0$ s.t.

$(\lambda_0 - A) D(A)$ is dense in X.

Then $\bar{A}$ generates a contractive

C_0-semigroup.

Proof. $\bar{A}$ is dissipative and dd.

Let $y \in X$. $\exists\, x_n \in D(A)$ $\lambda_0 x_n - Ax_n \to y$

$$\|\lambda_0(x_n - x_m)\| \leq \|\lambda_0(x_n - x_m) - A(x_n - x_m)\|$$
$$\longrightarrow 0 \quad \text{as } n, m \to \infty.$$

Thus $x := \lim x_n$ exists.

$\Rightarrow$ $\cancel{\lambda_0 x_n - Ax_n \longrightarrow}$

$$Ax_n = -(\lambda_0 x_n - Ax_n) + \lambda_0 x_n$$
$$\longrightarrow -y + \lambda_0 x$$

$\Rightarrow$ $x \in D(\bar{A})$ & $\bar{A}x = -y + \lambda_0 x$

$\lambda_0 x - \bar{A}x = y$. $\square$