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8. Elliptic operators

$\Omega \subset \mathbb{R}^d$ open.

(8.1) Weak derivatives

Let $u \in L^1_{loc}(\Omega)$

a) Let $j \in \{1, \dots d\}$, $f \in L^1_{loc}(\Omega)$

$D_j u = f \quad :\Longleftrightarrow$

$$-\int_\Omega \partial_j \varphi \, u = \int_\Omega f \varphi \qquad \forall \varphi \in C_c^\infty(\Omega)$$

b) Let $j \in \{1, \dots, d\}$; $E \subset L^1_{loc}(\Omega)$

$D_j u \in E \quad :\Leftrightarrow \quad \exists f \in E \;\; D_j u = f.$

$D_j u =:$ the weak derivative of f

Rk (consistence) Let $u \in C^1(\Omega)$. Then
$\partial_j u = D_j u$.

(8.2) Weak Laplacian. $u \in L^1_{loc}(\Omega)$

a) $f \in L^1_{loc}(\Omega)$

$$\Delta u = f \quad :\Leftrightarrow \quad \int_\Omega u \, \Delta \varphi = \int_\Omega f \varphi$$

$$\forall \varphi \in C_c^\infty(\Omega)$$

b) $\Delta u \in E \quad :\Leftrightarrow \quad \exists f \in E \quad \Delta u = f.$

(8.3) Sobolev space. $H^1(\Omega) := \{ u \in L^2(\Omega) : D_j u \in L^2(\Omega) \}$ is a Hilbert space

for

$$(u \mid v)_{H^1} = \int_\Omega u \bar{v} + \sum_{j=1}^{d} \int D_j u \, \overline{D_j v}.$$

Thus

$$\|u\|^2_{H^1} = \|u\|^2_{L^2} + \sum_{j=1}^{d} \|D_j u\|^2_{L^2}$$

Rk (convergence) $u_n \to u$ in $H^1(\Omega)$

$\Leftrightarrow$ $u_n \to u$ & $D_j u_n \to D_j u$

in $L^2(\Omega)$ as $n \to \infty$ $\quad j = 1, \dots, d.$

$$H^1_0(\Omega) := \overline{C_c^\infty(\Omega)}^{\cancel{L^2(\Omega)}\, H^1(\Omega)} .$$

<u>Definition</u> <u>(Dirichlet Laplacian).</u>

Let $D(\Delta^D) := \{ u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega) \}$

$\Delta^D u := \Delta u .$

<u>Theorem.</u> Δ^D generates a contractive C_0-semigroup T on $L^2(\Omega)$.

$\mathbb{K} = \mathbb{R}$

<u>Proof.</u> a) Δ^D is dissipative.

$$(\Delta^D u \mid u) = \int_\Omega (\Delta u) u \, dx$$

$$\int \Delta u \, \varphi = \cancel{\sum_j \int D} \int u \, \Delta \varphi$$

$$= \sum_j \int u \, D_j D_j \varphi$$

$$= - \sum_j \int D_j u \, D_j \varphi \qquad \varphi \in C_c^\infty(\Omega)$$

$C_c^\infty(\Omega)$ dense in $H_0^1(\Omega)$ $\Rightarrow$

$$\int \Delta u \, u = - \sum_{j=1}^{d} \int |D_j u|^2 \, dx \leq 0$$

$\forall u \in D(\Delta^D)$.

b) range condition : $\lambda_0 = 1$.

Let $f \in L^2(\Omega)$. Find $u \in H_0^1(\Omega)$ s.t.

$u - \Delta u = f$.

$L\varphi = \int_\Omega f \varphi \, dx$ defines $L \in (H_0^1(\Omega))'$

Riesz - Fréchet $\Rightarrow$ $\exists! \, u \in H_0^1(\Omega)$

$(u \mid \varphi)_{H^1} = L\varphi \quad \forall \varphi \in H_0^1(\Omega)$; i.e.

$$\int_\Omega u\varphi + \sum_{j=1}^{d} \int D_j u \, D_j \varphi = \int_\Omega f \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega)$$

$\Leftrightarrow \forall \varphi \in C_c^\infty(\Omega)$

$$\Rightarrow \int_\Omega u\varphi - \int_\Omega u \, \Delta\varphi = \int_\Omega f\varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

$\Rightarrow u - \Delta u = f$. $\square$

8.5

Let $a_{ij} \in L^\infty(\Omega)$ s.t.

$$\sum a_{ij}(x)\xi_i \xi_j \geq \alpha |\xi|^2 \qquad \forall \xi \in \mathbb{R}^d$$

a.e.

$$D(A) := \{ u \in H_0^1(\Omega) \quad \sum_{i,j=1}^{d} D_i a_{ij} D_j u \in L^2(\Omega) \}$$

$$Au = \sum_{i,j=1}^{d} D_i (a_{ij} D_j u)$$

Hier : $\displaystyle\sum_{i,j=1}^{d} D_i (a_{ij} D_j u) \in L^2(\Omega)$

$:\Leftrightarrow \; \exists f \in L^2(\Omega)$

$$-\int \sum_i \sum_{j=1}^{d} a_{ij} D_j u \; \partial_i \varphi \, dx = \int_\Omega f \varphi \, dx$$

$\forall \varphi \in \mathcal{D}(\Omega) := C_c^\infty(\Omega)$.

<u>Theorem.</u> A generates a contractive C_0-semigroup on $L^2(\Omega)$.

We assume here $a_{ij} = a_{ji}$.

ÜA <u>Proof.</u> a) dissipative: $u \in D(A)$

$$(Au \mid \varphi) = -\sum_i \int \sum_j a_{ij} D_j u \, \partial_i \varphi \, dx$$

$\forall \varphi \in \mathcal{D}(\Omega) \Longrightarrow$

$$(Au \mid u) = -\sum_i \int \sum_j a_{ij} D_j u \, D_i u \, dx$$

$$\leq 0 \qquad \forall u \in D(A).$$

b) ~~Let~~ $[u, v] = \alpha \int uv \, dx + \int \sum_{i,j=1}^{d} a_{ij}(x) D_i u \, D_j v \, dx$

defines an equivalent scalar product on ~~$L^2(\Omega)$~~ $H^1_0(\Omega)$. In fact

$$[u, u] \geq \int_\Omega |u|^2 dx + \alpha \int_\Omega |\nabla u|^2 d$$

$$|[u, v]| \leq \|u\|_{L^2} \|v\|_{L^2} + c \sum_{j=1}^{d} \| D_j u \| \, \| D_j v \|$$

$$\le \tfrac{1}{2}$$

$$\|u\|_{H^1}^2 \|v\|_{H^1}^2 = \Big(\|u\|_{L^2}^2 + \sum_j \|D_j u\|^2\Big) \cdot \Big(\|v\|_{L^2}^2 + \sum_j \|D_j v\|^2\Big)$$

$$\ge |[u, v]|.$$

Thus $\quad \alpha \|u\|_{H^1}^2 \le |[u, u]| \le C \|u\|_{H^1}^2$.

Let $f \in L^2(\Omega)$, $\quad L\varphi = \int_\Omega f\varphi$

Riesz–Fréchet $\Rightarrow \exists!\, u \in H^1_0(\Omega)$

$$[u, v] = \int f v \qquad \forall v \in H^1_0(\Omega)$$

$$\Rightarrow \int u\varphi + \int \sum_i \sum_j a_{ij} D_i u \, D_j \varphi = \int f \varphi$$

$$\forall \varphi \in \mathcal{D}(\Omega)$$

$$\Rightarrow \quad u - Au = f \quad , \; u \in D(A). \qquad \square$$

<u>Rk.</u> If the a_{ij} are not symmetric one uses the Lax Milgram Lemma instead of Riesz–Fréchet.

9 Adjoint ~~& the surjective LP Thm.~~

Definition. Let A be a d.d operator on X, X' the dual space of X. The adjoint A' of A is defined by

$$D(A') = \{ x' \in X' : \exists y' \in X' \quad \langle Ax, x' \rangle = \langle x, y' \rangle \ \forall x \in D(A) \}$$

$A'x' = y'$. Rk. Clearly: $(\lambda - A)' = \lambda - A'$.

(9.1) Lemma. Let $\lambda \in \rho(A)$. Then $\lambda \in \rho(A')$ and

$$R(\lambda, A') = R(\lambda, A)'$$

ÜA Proof. Let $x' \in X'$. Claim $R(\lambda, A)'x' \in D(A')$ and $(\lambda - A')R(\lambda, A)'x' = x'$.

$$\langle R(\lambda, A)'x', (\lambda - A)x \rangle = \langle x', R(\lambda, A)(\lambda - A)x \rangle$$
$$= \langle x', x \rangle \quad \forall x \in D(A)$$

$\Rightarrow \quad R(\lambda, A)'x' \in D((\lambda - A)')$ and

$(\lambda - A)' R(\lambda, A)' x' = x'$.

Conversely, let $x' \in D((\lambda - A)') = D(A')$

$\langle R(\lambda, A)'(\lambda - A')x', y\rangle = \langle (\lambda - A)'x', R(\lambda, A)y\rangle$

$= \langle x', (\lambda - A)R(\lambda, A)y\rangle = \langle x', y\rangle \quad \forall y \in X.$

$\Rightarrow \quad R(\lambda, A)'(\lambda - A')x' = x'. \; \square$

(9.2) Theorem. Let X be reflexive, A the generator of a C_0-sg T on X. Then $(T(t)')_{t\geq 0}$ is a C_0-sg and A' its generator.

Rk. Clearly, $(T(t)')_{t\geq 0}$ is a sg.

Pb.: strong continuity.

Proof. Replacing A by $A - \omega$ we may assume $\|T(t)\| \leq M$.

Passing to an equivalent norm we may assume $\|T(t)\| \le 1$.

$\Rightarrow \quad \|\lambda R(\lambda, A)\| \le 1$

$= \quad \|\lambda R(\lambda, A')\| \le 1$

$D(A')$ is dense: ~~$\lambda R(\lambda$,~~ Let

$x \in D(A)$ s.t. $\langle x, x' \rangle \quad \forall x' \in D(A')$

$\Rightarrow \quad \langle x, R(\lambda, A)' x' \rangle = 0 \quad \forall x' \in X'$.

$\Rightarrow \quad \langle R(\lambda, A) x, x' \rangle = 0 \quad \forall x' \in X'$

$\Rightarrow \quad R(\lambda, A) x = 0 \quad \Rightarrow x = 0$.

Rk (9.3) $\Rightarrow$ $D(A')$ is dense in X'.

LP $\Rightarrow$ A' generates a C_0-sg. $(S(t))_{t \ge 0}$ on X'. ~~Let $x' \in D(A')$~~

For $x \in X$, $x' \in X'$, $\lambda > 0$

$$\langle R(\lambda, A) x, x' \rangle = \int_0^\infty e^{-\lambda t} \langle T(t) x, x' \rangle \, dt$$

$$\|$$

$$\langle x, R(\lambda, A') x' \rangle = \int_0^\infty e^{-\lambda t} \langle x, S(t) x' \rangle \, dt$$

$\underset{\Longrightarrow}{\text{uniqueness Thm}} \quad \langle T(t) x, x' \rangle = \langle x, S(t) x' \rangle$

$\forall t > 0 \quad \Rightarrow \quad T(t)' = S(t)$. $\square$

<u>Uniqueness Theorem.</u> Let $f : (0,\infty) \to \mathbb{K}$ be measurable & bounded. If $\exists\, \lambda_0 \geq 0$ s.t.

$$\int_0^\infty e^{-\lambda t} f(t)\, dt = 0 \quad \forall\, \lambda > \lambda_0,$$

then $f(t) = 0$ a.e.

$\downarrow$

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