 15.5.2017

§10 The surjective LP Theorem

(10.1) **Lemma (kernel)** $\lambda \in \rho(A)$, $x \in X$.

Then

$$x \in \ker A \iff \lambda R(\lambda, A)x = x.$$

Pf. "$\Rightarrow$" $\lambda R(\lambda, A)x - R(\lambda, A)Ax = x$

"$\Leftarrow$" $\lambda R(\lambda, A)x \cancel{- AR(\lambda, A)x} = x$

Thus $\Rightarrow$ $x \in D(A)$ & $\lambda x = (\lambda - A)x$

$\Rightarrow Ax = 0$. $\square$

(10.2) **Recall**: **w^*-convergence.**

a) $x_n', x' \in X'$

$x_n' \xrightarrow{*} x'$ $:\iff$ $\langle x_n', x\rangle \to \langle x', x\rangle$ $\forall x \in X$

$\Rightarrow$ $\sup \|x_n'\| < \infty$.

b) <u>Theorem</u> <u>(Alaoglu Bourbaki)</u>.

X separable, $\|x_n'\| \le c$

$\Rightarrow \exists$ ss $\exists\, x' \in X'$ $\quad x_{n_k}' \xrightarrow{*} x'$.

c) $\quad S \in \mathcal{L}(X)$

$x_n' \xrightarrow{*} x' \quad \Rightarrow \quad S'x_n' \xrightarrow{*} S'x'$

<u>Pf</u>. $\langle S'x_n', x \rangle = \langle x_n', Sx \rangle \to \langle x', Sx \rangle$
$= \langle S'x', x \rangle$.

(10.3) <u>Theorem</u> <u>(LP: surjective version)</u>.

Let A be diss., dd, surj.
Then A is m-diss. & $0 \in \rho(A)$.

(10.4) <u>Theorem</u>. $\mathcal{L}_s(X, Y) := \{ S \in \mathcal{L}(X, Y)$ surjective$\}$ is open in $\mathcal{L}(X, Y)$

(10.5) <u>Kernel - separation lemma.</u>

Let A be ~~an~~ dd operator such that $\| \lambda R(\lambda, A) \| \leq M$ $\lambda \in (0, \delta]$, $\delta > 0$.

Let $x \in \ker A$, $x \neq 0$. Then $\exists x' \in \ker A'$, $\langle x', x \rangle \neq 0$

Pf. Assume that X is separable (for convenience) otherwise nets.

Let $x \in \cancel{X} \ker A$, $x \neq 0$

Let $x_0' \in X'$ $\langle x_0', x \rangle = 0$

$\exists \lambda_n \downarrow 0$ $\exists x' \in X'$ $\lambda_n R(\lambda_n, A)' x_0' \xrightarrow{*} x'$

$$\langle x', x_0' \rangle = \lim \langle \lambda_n R(\lambda_n, A)' x_0', x^* \rangle$$
$$= \lim \langle x_0', \lambda_n R(\lambda_n, A) x \rangle$$
$$= \lim \langle x_0', x \rangle = \langle x_0', x \rangle \neq 0$$

Thus $x' \neq 0$. Let $\mu \in \varrho(A)$ $\Rightarrow$

$$\lambda_n R(\mu, A)' R(\lambda_n, A)' x_0' \xrightarrow{*} R(\mu, A)' x'$$
$$\|$$
$$\frac{\lambda_n}{\mu - \lambda_n} R(\mu, A)' x_0' - \frac{\lambda_n}{\mu - \lambda_n} R(\lambda_n, A)' x_0'$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$0 \qquad - \frac{1}{\mu} x' \quad (10.1) \Rightarrow \text{claim}$$

□

Pf of (10.3). $\bar{A}$ is surj. & diss.

$\bar{A} \in \mathcal{L}(D(\bar{A}), X)$ surj. $\overset{(10.4)}{\Longrightarrow}$ $\exists \lambda > 0$

$\lambda - \bar{A}$ is surj. $\Rightarrow$ $\bar{A}$ is m-diss.

Assume $\exists x_0 \in \ker \bar{A}$, $x_0 \neq 0$.

$\overset{(10.5)}{\Longrightarrow}$ $\exists x_0' \in \ker \bar{A}'$, $x_0' \neq 0$.

$\Rightarrow$ $\langle Ax, x_0' \rangle = 0$ $\forall x \in D(A)$

A surj. $\Rightarrow$ $x_0' = 0$ ↯.

Thus $\bar{A}$ is bij. $\Rightarrow$ $0 \in \rho(\bar{A})$.

Let $x \in D(\bar{A})$. $\exists x_0 \in D(A)$

$Ax_0 = \bar{A}x$ $\Rightarrow$ $x_0 - x \in \ker \bar{A} = \{0\}$

$\Rightarrow$ $x = x_0 \in D(A)$. $\square$

Supplements to Alaoglu.

Definition (nets). Let $(I, \preceq)$ be an ordered set which is directed, i.e.
$\forall i_1, i_2 \in I \ \exists i_3 \in I \quad i_1 \preceq i_3, \ i_2 \preceq i_3.$

A family $(x_i)_{i \in I}$ is called a net.

Let $x \in X$, $\lim\limits_{I} x_i = x :\Leftrightarrow$
$\forall \varepsilon > 0 \ \exists i_0 \quad \|x - x_i\| < \varepsilon;$

~~$w^*-\lim x_i$~~

Let $x_i' \in X'$, $x' \in X'$

$w^*-\lim\limits_{I} x_i' = x' \Leftrightarrow \lim\limits_{I} \langle x_i', x\rangle = \langle x', x\rangle.$

$\Leftrightarrow \ \forall \varepsilon > 0 \ \exists i_0 \ \forall i \succeq i_0$
$|\langle x_i', x\rangle - \langle x', x\rangle| \le \varepsilon.$

Theorem (Alaoglu). Let X be a Banach space. Each bounded net in X' has a w^*-convergent subnet.

<u>Definition.</u> Let $(x_i)_{i\in I}$ be a net.

Let J be directed, $\phi: J \to I$ s.t.

(a) $j_1 \preccurlyeq j_2 \Rightarrow \phi(j_1) \preccurlyeq \phi(j_2)$

(b) $\forall i \in I \quad \exists j \in J \quad \phi(j) \succcurlyeq i$.

Then $(x_{\phi(j)})_{j\in J}$ is called a <u>subnet</u> of $(x_i)_{i\in I}$.

<u>Example:</u> $X = \ell^\infty$, $\langle e_n', x\rangle = x_n$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$.

Thus $e_n' \in X'$, $\|e_n'\| = 1$ $(n\in\mathbb{N})$

There is no w^*-convergent ~~subnet~~ subsequence of $(e_n')_{n\in\mathbb{N}}$.

Proof. Let $n_k < n_{k+1}$. Define $x \in \ell^\infty$ by
$$x_n = \begin{cases} (-1)^k & \text{if } n = n_k \\ 0 & \text{if } n \notin \{n_k : k\in\mathbb{N}\}. \end{cases}$$

Then $\langle e_{n_k}', x\rangle = (-1)^k$ does not converge $\square$

However, $(e_n')_{n\in\mathbb{N}}$ possesses a w^*-convergent subnet.

10. The Dirichlet Laplacian on $C_0(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be open, bded &

Dirichlet regular; i.e.

$\forall g \in C(\partial\Omega) \quad \exists u \in C(\bar{\Omega}) \cap C^2(\Omega)$

$$\Delta u = 0 \quad \text{in } \Omega$$

$$u_{|\partial\Omega} = g.$$

Example: a) Ω has Lipschitz boundary

b) $\Omega \subset \mathbb{R}^2$ is simply connected.

$$C_0(\Omega) := \{u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$$

Definition. The operator Δ_0 on $C_0(\Omega)$

given by

$$D(\Delta_0) := \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}$$

$$\Delta_0 u = \Delta u$$

is ~~of~~ called the <u>Dirichlet Laplacian</u>.

(10.1) <u>Theorem.</u> Δ_0 generates a contractive C_0-semigroup. T

Moreover, $T(t) \geq 0$ for all $t > 0$.

Let $0 < \rho \in \mathcal{D}(\mathbb{R}^d)$, $\int \rho = 1$,

supp $\rho \subset B(0,1)$. $\rho_n(x) = c_n \, \rho(nx)$

s.t. $\int \rho_n(x)\, dx = 1$. Then $\rho_n \in \mathcal{D}(\mathbb{R}^d)$,

supp $\rho_n \subset B(0, 1/n)$.

(11.2) <u>Lemma.</u> Let $f \in C(\mathbb{R}^d)$,

$$\rho_n * f(x) := \int_{|y| < \frac{1}{n}} f(y)\, \rho_n(x-y)\, dy \; .$$

Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$ &

$$\| \rho_n * f - f \|_{C(K)} \longrightarrow 0 \quad (n \to \infty)$$

$\forall K \subset \mathbb{R}^d$ compact.

<u>Proof.</u> a) $\partial_j (\rho_n * f) = \partial_j \rho_n * f$

b) $\rho_n * f(x) = \int\limits_{|y|<\frac{1}{n}} f(x-y)\, \rho_n(y)\, dy$.

Let $K_1 = \{ K + \overline{B}(0,1)$ compact.

Let $\varepsilon > 0$ $\quad \exists n_0 \quad |f(x-y) - f(x)| \leq \varepsilon$

$\forall x \in K, \ |y| \leq \frac{1}{n_0}$

$\Rightarrow |\rho_n * f(x) - f(x)| \leq \int |f(x-y) - f(x)| \rho_n(y)\, dy$

$\leq \varepsilon \qquad n \geq n_0 \qquad \square$

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(11.3) <u>Rk.</u> $v \in C^2(\Omega)$, $x_0 \in \Omega$ $\quad v(x_0) = \max\limits_{y \in \Omega} v(y)$

$\Rightarrow \Delta v(x_0) \leq 0$

<u>Pf.</u> $\sup\limits_{|t| \leq \delta} v(x_0 + t e_j) = v(x_0)$.

$\Rightarrow 0 \geq \frac{d^2}{dt^2}\Big|_{t=0} v(x_0 + t e_j) = \partial_j^2 v(x_0)$

$\Rightarrow \Delta v(x_0) = \sum\limits_{j=1}^{d} \partial_j^2 v(x_0) \leq 0 \qquad \square$