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10. The Dirichlet Laplacian on $C_0(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be open, bdded &

Dirichlet regular; i.e.

$\forall g \in C(\partial\Omega)$ $\exists u \in C(\bar{\Omega}) \cap C^2(\Omega)$

$$\Delta u = 0 \quad \text{in } \Omega$$

$$u_{|\partial\Omega} = g.$$

Example: a) Ω has Lipschitz boundary

b) $\Omega \subset \mathbb{R}^2$ is simply connected.

$$C_0(\Omega) := \{u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$$

Definition. The operator Δ_0 on $C_0(\Omega)$

given by

$$D(\Delta_0) := \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}$$

$$\Delta_0 u = \Delta u$$

is ~~ct~~ called the <u>Dirichlet Laplacian</u> on $C_0(\Omega)$.

(1~~0~~1.1) <u>Theorem.</u> Δ_0 generates a contractive C_0-semigroup T.
Moreover, $T(t) \geq 0$ for all $t > 0$.

Let $0 < \rho \in \mathcal{D}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \rho = 1$,
$\operatorname{supp} \rho \subset B(0,1)$. $\rho_n(x) = c_n \rho(nx)$
s.t. $\int \rho_n(x)\,dx = 1$. Then $\rho_n \in \mathcal{D}(\mathbb{R}^d)$,
$\operatorname{supp} \rho_n \subset B(0, 1/n)$.

(11.2) <u>Lemma.</u> Let $f \in C(\mathbb{R}^d)$,

$$\rho_n * f(x) := \int_{\mathbb{R}^d \cancel{|y|<\frac{1}{n}}} f(y)\, \rho_n(x-y)\,dy .$$

Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$ &

$$\| \rho_n * f - f \|_{C(K)} \longrightarrow 0 \quad (n \to \infty)$$

$\forall K \subset \mathbb{R}^d$ compact.

<u>Proof.</u> a) $\partial_j (\rho_n * f) = \partial_j \rho_n * f$

b) $\rho_n * f(x) = \int_{|y|<\frac{1}{n}} f(x-y)\, \rho_n(y)\, dy$.

Let $K_1 = \{ K + \overline{B}(0,1) \}$ compact.

Let $\varepsilon > 0$ $\exists n_0$ $|f(x-y) - f(x)| \le \varepsilon$

$\forall x \in K$, $|y| \le \frac{1}{n_0}$

$\Rightarrow |\rho_n * f(x) - f(x)| \le \int |f(x-y) - f(x)| \rho_n(y)\, dy$

$\le \varepsilon \quad n \ge n_0$ □

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(11.3) <u>Rk.</u> $v \in C^2(\Omega)$, $x_0 \in \Omega$ $v(x_0) = \max_{y \in \Omega} v(y)$

$\Rightarrow \Delta v(x_0) \le 0$

<u>Pf.</u> $\sup_{|t| \le \delta} v(x_0 + t e_j) = v(x_0)$.

$\Rightarrow 0 \ge \frac{d^2}{dt^2}\Big|_{t=0} v(x_0 + t e_j) = \partial_j^2 v(x_0)$

$\Rightarrow \Delta v(x_0) = \sum_{j=1}^{d} \partial_j^2 v(x_0) \le 0$ □

(11.4) Lemma. Let $u \in C_0(\Omega)$ such that

$$m := \max_{x \in \Omega} u(x) > 0.$$

Assume that $\Delta u \in C(\bar{\Omega})$.

Then $\exists\, x_0 \in \Omega$ s.t. $u(x_0) = m$ &

$\Delta u(x_0) \le 0$.

Proof. $C_0(\Omega) \subset C(\mathbb{R}^d)$

$u_n = u * \rho_n \longrightarrow u$ in $C(\bar{\Omega})$.

~~Let~~ ~~K.~~ Let $x_n \in \bar{\Omega}$ such that $u_n(x_n) = \max_{x \in \bar{\Omega}} u_n(x)$.

We may assume that $x_n \to x_0 \in \bar{\Omega}$

Since $u_n \to u$ uniformly,

$u_n(x_n) \to m$.

~~Observe that $u_n(x) = \int \rho_n(y) u(x-y)\,dy$~~

~~$\le m \quad \forall x \in \bar{\Omega},\ n \in \mathbb{N}$.~~

$$u(x_0) = \underbrace{(u(x_0) - u(x_n))}_{\downarrow \atop 0} + \underbrace{(u(x_n) - u_n(x_n))}_{\downarrow \atop 0} +$$

$$u_n(x_n) \longrightarrow m.$$

Thus $u(x_0) = m$.

<u>Claim</u>: $\Delta u_n(x_n) \longrightarrow \Delta u(x_0)$.

[In fact, let $n_0 \in \mathbb{N}$ such that $\overline{B}(x_n, \frac{1}{n_0}) \subset \Omega$ $\forall n \geq n_0$. Then for $n \geq n_0$ $\varphi(y) = \rho_n(x_n - y)$ defines $\varphi \in \mathcal{D}(\Omega)$. Thus

$$\Delta u_n(x_n) = \int \Delta \rho_n(x_n - y)\, u(y)\, dy$$

$$= \int \Delta\varphi(x_n - y)\, u(y)\, dy$$

$$= \int \varphi(x_n - y)\, \Delta u(y)\, dy$$

$$= (\rho_n * \Delta u)(x_n).$$

The proof of (11.2) shows that $\rho_n * \Delta u \longrightarrow \Delta u$ uniformly on compact subsets of Ω.

Thus

$$\Delta u_n(x_n) = (\rho_n * \Delta u)(x_n)$$

$$= (\rho_n * \Delta u)(x_n) - \Delta u(x_n) + \Delta u(x_n)$$

$$\longrightarrow \Delta u(x_0) \quad (n \to \infty).\ ]$$

By (11.3) $\Delta u_n(x_n) \leq 0$. Thus

$\Delta u(x_0) \leq 0$.

(11.5) <u>Lemma</u>. Let $u \in D(\Delta_0)$, $\lambda > 0$,

$\lambda u - \Delta_0 u = f$.

If $f(x) \leq 1$ $\forall x \in \Omega$, then

$\lambda u(x) \leq 1$ $\forall x \in \Omega$.

<u>Proof.</u> <u>1st case</u>: $u \leq 0$ trivial

<u>2nd case</u> $m = \sup_{x \in \Omega} u(x) > 0$.

(11.4) $\Rightarrow$ $\exists x_0 \in \Omega$ $u(x_0) = m$, $\Delta u(x_0) \leq 0$

$$\Rightarrow \lambda u(x_0) \leq \lambda u(x_0) \underbrace{- \Delta u(x_0)}_{\geq 0} = f(x_0) \leq 1$$

$$\Rightarrow \lambda u(x) \leq \lambda u(x_0) \leq 1 \quad \forall x \in \Omega. \quad \square$$

(11.6) <u>Lemma.</u> Δ_0 ist dissipativ.

<u>Beweis.</u> Sei $u \in D(\Delta_0)$, $\lambda > 0$

$\lambda u - \Delta u = f$.

$\|f\|_\infty = 1$. Claim $\lambda \|u\|_\infty \leq 1$.

<u>1st case:</u> $\exists x_0$ $f(x_0) = \|f\|_\infty$.

$\Rightarrow \lambda u(x) \leq 1 \quad \forall x \in \Omega$.

<u>2nd case</u> $\lambda u - \Delta u = f \geq -1$

$\Rightarrow \lambda(-u) - \Delta(-u) \leq +1$

$\Rightarrow -u \leq 1 \Rightarrow u \geq 1$.

~~At~~ Thus $\|\lambda u - \Delta u\|_\infty \leq 1 \Rightarrow \lambda \|u\|_\infty \leq 1$. □

(11.7) Fundamental solution of the Laplace equation.

$$E(x) = \begin{cases} \frac{1}{2}|x| & d=1 \\ \frac{1}{2\pi} \log|x| & d=2 \\ \frac{1}{-(d-2)\sigma_d} \frac{1}{|x|^{d-2}} & d \geq 3 \end{cases}$$

$$\sigma_d = |\partial B|$$

Then $E \in L^1_{loc}(\mathbb{R}^d)$ &

$$\Delta E = \delta_0$$

i.e. $\int_{\mathbb{R}^d} E \Delta\varphi = \varphi(0) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d)$

Fundamental solution:

Let $f \in C_c(\mathbb{R}^d)$

$$u = E * f$$

Then $u \in C^1(\mathbb{R}^d)$ &

$$\Delta u = f$$

(i.e $\int_{\mathbb{R}^d} u \Delta\varphi = \int_{\mathbb{R}^d} f\varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d)$)

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Proof of Theorem 11.1.

a) Δ_0 is dissipative.

b) $\mathcal{D}(\Omega) \subset D(\Delta_0)$ and $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$ $\Rightarrow$ ~~D(Δ~~ Δ_0 is d.d.

c) Δ_0 is surjective.

Let $f \in C_0(\Omega) \subset C_c(\mathbb{R}^d)$. $\Rightarrow$

$u = E * f \in C^1(\mathbb{R}^d)$ $\quad \Delta u = f$.

Let $g = u_{|\partial\Omega}$. $\exists\ w \in C^2(\Omega) \cap C(\bar{\Omega})$

$\Delta w = 0$, $w_{|\partial\Omega} = g$.

Let $v = u - w \in C_0(\Omega)$

Then $\Delta v = \Delta u - \Delta w = \Delta u = f$. $\square$

The surj. LP Theorem implies the claim. $\square$

(11.8) Lemma. $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$

Proof. Let $f \in C_0(\Omega)$, $\varepsilon > 0$

$K = \{x : |f(x)| \geq \varepsilon\} \subset \Omega$ compact.

Choose $\varphi \in \mathcal{D}(\Omega)$ such that

$\cancel{0 \le \varphi \le}$, $1_K \le \varphi \le 1_\Omega$.

Then $\varphi \cdot f \in C_c(\Omega)$

$|(f - \varphi \cdot f)(x)| = 0 \quad x \in K$

$|(f - \varphi f)(x)| \le \varepsilon\, |1-\varphi| \le \varepsilon \ (x \notin K)$.

Thus $C_c(\Omega)$ is dense in $C_0(\Omega)$.

b) Let $f \in C_c(\Omega)$. Let $f_n = \rho_n * f$.

Then $\operatorname{supp} f_n \subset \operatorname{supp} f + \operatorname{supp} \rho_n$

$\subset \operatorname{supp} f + B(0, \frac{1}{n})$

$\subset K$

Thus $f_n \in \mathcal{D}(\Omega)$ if $n \ge n_0$ $\square$

(11.9) Bemerkung. (11.6) $\Rightarrow$ ~~Δ_0 is~~

$\lambda R(\lambda, \Delta_0)$ is submarkovian; i.e.

$f \le 1 \Rightarrow \lambda R(\lambda, \Delta_0) f \le 1$

In particular, $f \ge 0 \Rightarrow R(\lambda, \Delta_0) f \ge 0$

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