10 (11.8) **Definition** ~~Rem.~~: $T \in \mathcal{L}(C_0(\Omega))$ **submarkovian** $:\Leftrightarrow$

$$f \le 1 \quad \Rightarrow \quad Tf \le 1$$

Consequence: a) $T \geq 0$, i.e. $f \geq 0 \Rightarrow Tf \geq 0$

b) T^k submarkovian $\forall k \in \mathbb{N}$

Proof. a) $f \le 0 \Rightarrow f \le \lambda 1 \quad \forall \lambda > 0$

$\Rightarrow \frac{f}{\lambda} \le 1 \quad \forall \lambda > 0 \quad \Rightarrow \frac{1}{\lambda} Tf \le 1 \quad \forall \lambda > 0$

$\Rightarrow Tf \le 0.$

$f \geq 0 \Rightarrow -f \le 0 \Rightarrow -Tf \le 0 \Rightarrow Tf \geq 0.$ □

(11.9) **Proposition**. Let T be a C_0-semigroup on $C_0(\Omega)$ with generator A.

Equivalent:

(i) T is submarkovian

(ii) $\exists \lambda_0 > \omega$ $\lambda R(\lambda, A)$ submarkovian $\forall \lambda \geq \lambda_0$

(i) $\Rightarrow$ (ii) $\qquad \|T(t)\| \le M e^{\omega t}$

$\lambda > \omega \qquad \lambda R(\lambda, A) f = \int_0^\infty \lambda e^{-\lambda t} T(t) f \, dt$

$f \le 1 \Rightarrow T(t) f \le 1 \Rightarrow$

$\lambda R(\lambda, A) f \le \int_0^\infty \lambda e^{-\lambda t} dt = 1.$

(ii) $\Rightarrow$ (i) $\qquad T(t) f = \lim e^{tA_n} f$

e^{tA_n} submarkovian?

$A_n = n(nR(n, A) - I)$

$f \le 1 \Rightarrow (nR(n, A))^k \le 1$

$\Rightarrow \quad e^{tA_n} = e^{-nt} \sum \frac{t^k n^k}{k!} (nR(n, A))^k f$

$\le 1.$ □

Conclusion.

(11.10) <u>Theorem</u>. The semigroup T generated by Δ_0 is submarkovian.

§ 12 Perturbation

(12.1) Example. Let $B \in \mathcal{L}(X)$. Then

$B - \|B\| I$ is diss.

Proof.

$$\|e^{tA}\| = e^{-t\|B\|} \|e^{tB}\| = e^{-t\|B\|} \left\| \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \right\| \leq 1. \quad \square$$

(12.2) Proposition. A m-diss., $B \in \mathcal{L}(X)$ diss

$\Rightarrow$ $A + B$ m-diss.

If $\overline{D(A)} = X$, then $A+B$ gen. a contractive C_0-sg.

Pf. a) Let $x \in D(A)$. $\exists x' \in J(x)$

$\mathrm{Re} \langle Ax, x' \rangle \leq 0$. (7.10) $\Rightarrow$

B is strictly m-diss $\Rightarrow$

$\mathrm{Re} \langle Bx, x' \rangle \leq 0 \Rightarrow$

$\mathrm{Re} \langle Ax + Bx, x' \rangle \leq 0$.

Thus $A + B$ diss.

b) Let $\lambda > 0$, $\lambda > \|B\|$

$$\lambda - A - B = (I - BR(\lambda, A))(\lambda - A).$$

$$\Rightarrow \|BR(\lambda, A)\| \leq \frac{\|B\|}{\lambda} < 1$$

$\Rightarrow \lambda - A - B$ surj. □

(12.3) <u>Theorem.</u> A generator of a C_0-sg T, $B \in \mathcal{L}(X) \Rightarrow A + B$ generator of a C_0-sg.

<u>Proof.</u> a) First case: $\|T(t)\| \leq M$.

Then $\|T(t)\|_0 \leq 1$ for equ. norm.

(12.2) $\Rightarrow$ $A + B - \|B\|_0$ generates a contraction s.g. $\Rightarrow$ $A + B$ gen. a sg. S

$\|S(t)\|_0 \leq e^{\|B\|_0 t}$

<u>Second case</u> $\|T(t)\| \le Me^{\omega t}$

1st case $\Rightarrow$ $A - \omega + B$ generates a

C_0-sg $\Rightarrow$ $A + B$ generator. □

(12.4) <u>Exercise:</u> A generator of T on X

$\mathcal{U} : X \to Y$ isomorphism.

$\Rightarrow$ $S(t) = \mathcal{U}\, T(t)\, \mathcal{U}^{-1}$ C_0-sg

Generator: $D(B) = \{ y \in Y : \mathcal{U}^{-1} y \in D(A) \}$

$By = \mathcal{U} A \mathcal{U}^{-1} y$.

(12.5) <u>Exercise.</u> A generator of T on X.

$D(A)$ with graph norm

Then $T_1(t) = T(t)_{|D(A)}$ is a C_0-sg.

Its generator is A_1 given by

$D(A_1) = D(A^2)$

$A_1 x = Ax$.

(12.6) Exercise. (converse of 12.5).

Let A be an operator on X with $\varrho(A) \neq \emptyset$. If A_1 generates a C_0-sg on $D(A)$, then A generates a C_0-sg on X

(12.7) Theorem. Let A be the gen. of a C_0-sg on X, $B \in \mathcal{L}(D(A))$. Then $A+B$ generates a C_0-sg on X

Proof. $A_1 + B$ generates a C_0-sg on $D(A)$.

~~It suffices~~ Obviously, $A_1 + B = (A+B)_1$

It suffices to show that $\varrho((A+B)_1) \neq \emptyset$

$\exists\, \lambda_0^{\omega, M}$ s.t. $\|R(\lambda, A_1)\| \leq M \frac{1}{\lambda - \omega}$. $(\lambda > \omega)$

Choose $\lambda_0 > \omega$ s.t.

$$\|R(\lambda_0, A_1)\|_{\mathcal{L}(D(A))} \|B\|_{\mathcal{L}(D(A))} < 1$$

Then $\quad (I - R(\lambda_0, A_1)B_1) : D(A) \to D(A)$

is ~~bijective~~ invertible. $\Rightarrow$

$$(\lambda_0 - A - B) = (\lambda_0 - A)(I - R(\lambda_0, A_1)B) :$$

$D(A) \to X$ is bijective. and

$$(\lambda_0 - A - B) = (I - R(\lambda_0, A_1)B)^{-1} R(\lambda_0, A)$$

$\in \mathcal{L}(X, D(A)) \subset \mathcal{L}(X)$. Thus $\lambda_0 \in \rho(A)$.

$\square$

(12.8) <u>Definition.</u> Let A be a closed operator. $D \subset D(A)$ <u>core</u> $:\Leftrightarrow$

$\overline{A_{|D}} = A$

$\Leftrightarrow$ D is dense in $(D(A), \| \|_A)$.

(12.9) <u>Proposition</u> <u>(uniqueness)</u>

Let A be generator of T, B of S., $D \subset D(A)$ a core of A.

$A_{|D} \subset B \Rightarrow A = B$ & $S(t) = T(t)$

<u>Proof.</u> $A_{|D} \subset B$ & $\Rightarrow A = \overline{A_{|D}} \subset B$.

Let $x \in D(A)$, $u(t) = T(t)x$ solves

$\dot{u}(t) = Au(t) = Bu(t)$ $\Rightarrow$ $u(t) = S(t)x$.

$u(0) = x_0$

Thus $T(t)x = S(t)x$ $\forall x \in D(A)$

$\overline{D(A)} = X \Rightarrow T(t) = S(t)$.

(12.10) <u>Theorem.</u> T sg with generator A.

$D_0 \subset D(A)$ not a core. $\Rightarrow$

$\exists \infty$ many generators extending

$A_0 = A|_{D_0}$

<u>Pf.</u> ~~Let~~ HB $\exists \varphi \in D_0^{\perp}\ D(A)'$

$\varphi \neq 0$, $\varphi|_{D_0} = 0$ $\quad Bx = \langle \varphi, x \rangle u$

$u \in D(A)$ $\quad A + B$ Generator

$(A+B)|_{D_0} = A_0$ $\quad \square$

(12.11) <u>Proposition.</u> A Generator of T,

$D_0 \subset D(A)$, $\quad \overline{D_0}^X = X$

$T(t) D_0 \subset D_0$

$\Rightarrow$ D_0 is a core.

<u>Proof.</u> Let ~~D_0~~ $A_0 \subset B$ generator of S

~~mit~~ Let $x_0 \in D_0$, $\quad u(t) = T(t)x$ sol.

of $\quad \dot{u}(t) = Bu(t)$ $\quad \Rightarrow u(t) = S(t)x_0$

$\quad u(0) = x_0$ $\quad \Rightarrow S(t) = T(t)$ on D_0

$\Rightarrow S \equiv T$. $\square$