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§ 13 Selfadjoint operators

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(13.1) <u>Remark</u>

Let $S \in \mathcal{L}(H)$

a) $\exists!\ S^* \in \mathcal{L}(H)$ s.t.

$(Sx \mid y) = (x \mid S^* y) \quad \forall x, y \in H$

b) S s.a. $:\Leftrightarrow S = S^*$

$\Leftrightarrow (Sx \mid y) = (x \mid Sy)$

$\forall x, y \in H$. $:\Leftrightarrow$ S symmetric

(13.2) <u>Definition</u>. A operator on H, $\overline{D(A)} = H$

a) $D(A^*) = \{ y \in H : \exists z \in H$

$(Ax \mid y) = (x \mid z) \quad \forall x \in D(A) \}$

$A^* y = z$.

Thus $(Ax \mid y) = (x \mid A^* y) \quad \forall x \in D(A), y \in D(A^*)$

A^* is the <u>adjoint</u> of A.

b) A is <u>symmetric</u> if

$$(Ax|y) = (x|Ay) \qquad \forall x, y \in D(A)$$

~~Remark. A A generator of T a) ⟹ A* generator of T*(t)~~
~~b) A = A symmetric ⟺ T(t) = T(t)*~~

(13.3) <u>Lemma.</u> A dd. Then

(i) A sym.

$\Updownarrow$

(ii) $A \subset A^*$

<u>Proof.</u> (i) ⟹ (ii) $y \in D(A)$ ⟹

$$(Ax|y) = (x|Ay) \qquad \forall x \in D(A)$$

$$\Rightarrow \quad y \in D(A^*) \;\&\; A^*y = Ay.$$

(ii) ⟹ (i) Let $y \in D(A)$. Then $y \in D(A^*)$ & $A^*y = Ay$ ⟹ $\forall x \in D(A)$

$$(Ax|y) = (x|A^*y) = (x|Ay). \square$$

(13.4) <u>Lemma.</u> Let A be dd. Then A^* is closed.

<u>Pf.</u> $y_n \in D(A^*)$ $y_n \to y$, $A^*y_n \to z$

$x \in D(A)$ $\quad (Ax|y_n) = (x|A^*y_n) \to (x|z)$

$$\downarrow$$

$$(Ax|y)$$

$$\Rightarrow y \in D(A^*) \quad A^*y = z. \square$$

(13.5) Corollary. A dd sym. $\Rightarrow$

A closable

Pf. $A \subset A^*$, A^* closed. $\square$

(13.6) Polarization identity. V ~~unitary~~ vector space, $\mathbb{K} = \mathbb{C}$.

$a : V \times V \to \mathbb{C}$ sesquilinear; i.e.

$a(\cdot, y) : V \to \mathbb{C}$ linear,

$a(x, \cdot) : V \to \mathbb{C}$ antilinear.

Here $\varphi : V \to \mathbb{C}$ antilinear $:\Leftrightarrow \overline{\varphi}$ linear.

$a(x) := a(x, x)$

$$a(x,y) = \frac{1}{4}\left\{ a(x+y) - a(x-y) + i\,a(x+iy) - i\,a(x-iy)\right\}$$

Corollary. a symmetric $:\Leftrightarrow$

$a(x,y) = \overline{a(y,x)} \Leftrightarrow$

$a(x) \in \mathbb{R} \quad \forall x \in V$.

~~(13.7) Corollary. A operator on H, $\mathbb{K} = \mathbb{C}$~~

~~A symmetric $\Leftrightarrow (Ax \mid x) \in \mathbb{R}$ $\forall x \in D(A)$.~~

<u>Definition</u>. An operator A is symmetric if

$$(Ax \mid y) = (x \mid Ay) \quad \forall x, y \in D(A).$$

$\mathbb{K} = \mathbb{C}$

(13.7) <u>Corollary</u>. Equivalent

(i) A symmetric;

(ii) $(Ax \mid x) \in \mathbb{R} \quad \forall x \in D(A)$;

(iii) $\pm iA$ dissipative.

(13.8) <u>Proposition</u>. Let T be a C_0-sg with ~~Equ.~~ generator A.

a) $(T(t)^*)_{t \geq 0}$ is a sg and A^* its generator;

b) Equivalent:

(i) $T(t) = T(t)^* \quad \forall t \geq 0$

(ii) $A = A^*$

(iii) A symmetric

Pf. b) (iii) $\Rightarrow$ (ii)

A sym. $\rightarrow$ $A \subset A^*$

$\Rightarrow$ $A = A^*$ since both are generators.

(13.9) Definition. unitary group :=

C_0-group of unitary operators

$(\mathcal{U}(t))_{t \in \mathbb{R}}$

Consequence: 1. $\mathcal{U}(t)^* = \mathcal{U}(-t)$.

2. B the generator $\Rightarrow$ $-B = B^*$.

(13.10) Theorem. Let A be an operator on H.

Equ: (i) iA generates a unitary group

(ii) a) A symmetric

b) $\pm i - A$ surj. (range condition)

(iii) A dd & $A = A^*$

$\Leftrightarrow$: (A is selfadjoint)

(iv) $\pm iA$ is m-dissipative.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (i') clear.

(ii) $\Rightarrow$ (iii) 1. $A \subset A^*$

2. $i - A^*$ injective:

$ix - A^*x = 0 \Rightarrow$

$(iy + Ay \mid x) = (y \mid -ix + A^*x) = 0$

$\forall y \in D(A)$. But $R(i+A) = H \Rightarrow x = 0$

3. $\pm iA$ is m-diss. $\Rightarrow$ dd

4. Let $x \in D(A^*)$ $\exists y \in D(A)$

$iy - Ay = ix - A^*x$, $u := x - y$

$(i - A^*)u = 0 \Rightarrow u = 0 \Rightarrow x = y \in D(A)$.

(iii) $\Rightarrow$ (ii) $A = A^* \Rightarrow A$ closed & symmetric

$\Rightarrow \pm iA$ diss

$\Rightarrow \| \pm ix - Ax \| \geq \| x \|$ (1)

$\Rightarrow R(\pm i - A)$ closed.

Suppose $R(\pm i - A) \neq H$

$\Rightarrow \exists z \neq 0 \quad (\pm ix - Ax \mid z) = 0$

$\forall x \in D(A)$

$\Rightarrow \mp iz - Az = 0$

$\stackrel{(1)}{\Rightarrow} z = 0$. $\square$

$\boxed{\text{ÜA}}$ A s.a. $\Longleftrightarrow$

A dd, symmetric, closed and

$\pm i - A^*$ surjective

Equi:

(13.11) <u>Corollary</u>. A dd - symmetric

(i) $\bar{A}$ is sa

(ii) $(\pm i - A)D(A)$ is dense

<u>Pf.</u> $\boxed{\text{ÜA}}$.

<u>Rk.</u> A is <u>essentially s.a.</u> $:\Longleftrightarrow$ A dd, symmetric & $\bar{A}$ s.a.

(13.12) <u>Example</u> (multiplication operator)

(Ω, Σ, μ) measure space, $H = L^2(\Omega)$

$m : \Omega \to \mathbb{R}$ measurable

$A_m f = m f$

$D(A_m) = \{ f \in L^2 : mf \in L^2 \}$

Then A_m is selfadjoint.

$U(t) f = e^{itm} f$

defines a unitary C_0 group.

Its generator is iA_m.

(13.13) <u>Spectral Theorem.</u>

Let A be selfadjoint on H

Then $\exists\ \Phi : H \to L^2(\Omega, \Sigma, \mu)$

where (Ω, Σ, μ) is a measure space

& $\exists\ m : \Omega \to \mathbb{R}$ measurable

such that

$$\begin{array}{ccc} D(A) & \xrightarrow{A} & H \\ \downarrow & & \downarrow \Phi \\ D(A_m) & \xrightarrow{A_m} & L^2(\Omega, \Sigma, \mu) \end{array}$$

$$D(A_m) = \Phi D(A) \qquad \Phi^{-1} A_m \Phi = A.$$

Let $\mathcal{U}$ be the unitary group generated by iA. Then

$$\Phi\, \mathcal{U}(t) \Phi^{-1} g = e^{itm} g$$

$$\begin{tikzcd} H \arrow[r, "\mathcal{U}(t)"] \arrow[d, "\Phi"'] & H \arrow[d, "\Phi"] \\ L^2(\Omega) \arrow[r] & L^2(\Omega) \end{tikzcd}$$

$$g \longmapsto e^{itm} g.$$